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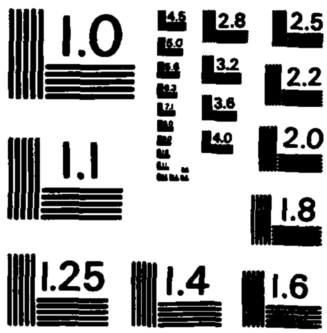

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Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS
ON THE DUAL OF A COUNTABLY HILBERT NUCLEAR SPACE
WITH APPLICATIONS TO NEUROPHYSIOLOGY

by

Søren Kier Christensen

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TECHNICAL REPORT 104

June 1985

85 9

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SECURITY CLASSIFICATION OF THIS PAGE

AD-A159 198

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT: unlimited. Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): Technical Report 104	5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 85-0705
6a. NAME OF PERFORMING ORGANIZATION: Center for Stochastic Processes; 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State and ZIP Code): Statistics Dept., 321 PH 039A, UNC, Chapel Hill, NC 27514	7b. ADDRESS (City, State and ZIP Code): AFOSR/NM, Bldg 410, Bolling AFB DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR; 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: F49620-82-C-0009
8c. ADDRESS (City, State and ZIP Code): Bolling AFB, Washington, DC 20332	10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	2304	A5	

11. TITLE (Include Security Classification): LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS ON THE DUAL OF A COUNTABLY HILBERT SPACE WITH APPLICATIONS TO NEUROPHYSIOLOGY

12. PERSONAL AUTHOR(S): Søren Kier Christensen

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Technical	FROM 9/84 TO 8/85	June 1985	212

16. SUPPLEMENTARY NOTATION

17. COSATI CODES		
FIELD	GROUP	SUB. GR.

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number): Dual of a countably Hilbert nuclear space, nuclear space valued Ornstein-Uhlenbeck process, existence and uniqueness of solutions to linear Φ'-valued stochastic differential equations, weak convergence of solutions.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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Existence and uniqueness of solutions to linear stochastic differential equations on the dual of a nuclear space is established, and general conditions for the weak convergence on Skorohod space of solutions are given. Moreover, solutions are shown to be CADLAG semimartingales (for appropriate initial conditions).

The results are applicable to solving stochastic partial differential equations.

Finally, the results are applied to giving a rigorous representation and solution of models in neurophysiology as well as to deriving explicit results for the weak convergence of these solutions.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL: Maj. Woodruff	22b. TELEPHONE NUMBER (Include Area Code): (202) 767-5027; 22c. OFFICE SYMBOL: NM

DD FORM 1473, 83 APR — EDITION OF 1 JAN 73 IS OBSOLETE.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

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This work supported in part by AFOSR Grant No. F49620 82 C 0009, the Egmont H. Petersens Fond of Copenhagen, Denmark, and by the Denmark-America Foundation.

SØREN KIER CHRISTENSEN. Linear Stochastic Differential Equations on the Dual of a Countably Hilbert Nuclear Space With Applications to Neurophysiology. (Under the direction of Gopinath Kallianpur.)

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ACKNOWLEDGEMENTS

I would like to extend my gratitude to my advisor, Professor Gopinath Kallianpur for introducing me to the topics considered here as well as for his guidance and stimulating attitude throughout its preparation.

I also wish to thank my committee members, Dr. Walter Laws Smith, Dr. David Ruppert, Dr. Stamatis Cambanis and in particular Dr. Charles R. Baker for their help.

My thanks are due to the entire Statistics Department for their support during my stay at the University of North Carolina at Chapel Hill. In particular, I wish to thank the secretarial staff for their good spirits and extraordinary skills at handling practical matters.

Special thanks are also due to my parents for their continued support throughout my education.

Finally, I am deeply indepted to my wife Catalina, not only for her endurance and encouragement, but also for her excellent work in typing this manuscript.

This work was supported in part by the Egmont H. Petersens Fond of Copenhagen, Denmark and by the Denmark-America Foundation.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION AND MOTIVATION

Within the last six years a number of publications concerning SDEs on the dual of a nuclear space have appeared. In a series of these articles [10], [11], [12], K. Itô has investigated special SDEs on the spaces $\mathcal{S}'$(= space of all tempered distributions) and $\mathcal{D}'$(= space of all distributions), and other authors have studied particular SDEs on more general dual nuclear spaces including Y. Miyahara [23], and G. Kallianpur & R. Wolpert [14].

Apart from its appealing probabilistic aspects research in this area has been stimulated by applications to such diverse fields as infinite particle systems in statistical mechanics (Holley and Stroock, [8]), chemical reaction kinetics (P. Kotelenez, [17]) and, most recently, to neurophysiology (G. Kallianpur & R. Wolpert, [14], [15]).

The primary motivation for studying SDEs on the dual of a nuclear space comes from the desire to solve stochastic partial differential equations (SPDEs). Here, we shall restrict attention to linear SPDEs.

Just as in the case of classical partial differential equations there are basically two different approaches to this problem:

<u>I:</u> Given a suitable partial differential operator (PDO) D in d dimensions and a Wiener process $W_{t,x}$ indexed by time $t \geq 0$ and spatial points $x \in \mathbb{R}^d$, find a process V (indexed by $t \geq 0$ and $x \in \mathbb{R}^d$) such that

$$dV(t,x) = DV(t,x)dt + dW_{t,x}$$

$$V(0,x) = V_o(x)$$

The main problem with this approach is that even for a very simple D a solution of this form may not exist (take for example $d = 2$ and $D = \quad - I$; see J.B. Walsh [29] section 10).

Therefore, inspired by the development of classical PDE theory, one may try to look for generalized solutions instead:

<u>II:</u> Given a suitable PDO D, a space Φ of "test functions" and a Φ'-valued Wiener process W (to be defined), find a Φ'-valued process $\eta = (\eta_t)_{t\geq 0}$ such that

$$\begin{aligned} d\eta_t[\phi] &= \eta_t[D\phi]dt + dW_t[\phi] \\ & \qquad\qquad\qquad\qquad \forall\ \phi \in \Phi \\ \eta_o[\phi] &= x[\phi] \end{aligned}$$

-Countably Hilbert nuclear spaces (see Appendix) were introduced by Gel'fand as generalizations of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and therefore seem appropriate as a choice for Φ.

Perhaps, one may wonder if it is not sufficient, for all practical purposes, to consider the case $\Phi = \mathcal{S}(\mathbb{R}^d)$. However, as pointed out in [14], this is far from the case; even in applications (such as neurophysiology, where a suitable Φ may be a space of infinitely differentiable functions on a compact Riemannian manifold) there is no guarantee that the relevant space of test functions can be accomodated as a subspace of $\mathcal{S}(\mathbb{R}^d)$.

Until now, no general theory has been developed for stochastic partial differential equations on the dual of a countably Hilbert nuclear space. Our primary objective here is to solve the following problems (see Appendix for terminology):

Let $\Phi \hookrightarrow H \hookrightarrow \Phi'$ be a rigged Hilbert space. Let $A : \Phi \rightarrow \Phi$ be linear and continuous. Let $M = (M_t)_{t \geq 0}$ be a Φ'-valued L^2-semimartingale (to be defined) and let η be a Φ'-valued random variable.

i) Give conditions on A assuring that the SDE on Φ'

$$d\xi_t = A'\xi_t dt + dM_t; \quad \xi_o = \eta$$

has a unique solution.

ii) The solution is, of course, a process on Φ'. But $\Phi' = {}_{q \geq 0}\Phi_{-q}$, and hence it is also relevant to ask whether for some $q \geq 0$ $_t \in \Phi_{-q}$ for all $t \in [0,\infty)$ or at least all $t \in [0,T]$ for some $T > 0$.

iii) Investigate the weak convergence of solutions; i.e. loosely speaking, if the noise and the initial condition converge weakly then does the solution also converge weakly ?

Chapter III, which is the main chapter, is devoted to the solution of these problems. In chapter IV we address our second objective which is to suggest a new approach to modelling neuronal behaviour via Φ'-valued SDEs, and to illustrate how the weak convergence result from chapter III can be useful in the context of modelling in neurophysiology.

Special examples of solutions to one particular class of Φ'-valued SDEs, namely the infinite-dimensional Ornstein–Uhlenbeck equations, have been subject to study by several authors ([23],[14],[8] and [29]), and therefore we shall commence by presenting a treatment of some of the properties of the general Ornstein–Uhlenbeck process on Φ' (chapter II).

For the convenience of the reader we include an Appendix presenting a definition and some basic properties of countably Hilbert nuclear spaces.

CHAPTER II

PROPERTIES OF THE Φ'-VALUED ORNSTEIN–UHLENBECK PROCESS

In this chapter we shall investigate some of the properties of the Ornstein–Uhlenbeck process on the dual of a countably Hilbert nuclear space (see Appendix for definition). Our interest in this particular process is aroused mainly by a paper by Miyahara [23] and by its recent applications in neurophysiology [14], [29].

However, the literature so far has dealt only with particular examples of these Ornstein–Uhlenbeck processes, and therefore a treatment of the general case seems appropriate. We shall discuss the issues of stationarity, absolute continuity of the transition measure wrt. to the invariant measure; and flicker noise.

However, first we must introduce some terminology:

II.1. PRELIMINARIES AND NOTATION

Let H be a real separable Hilbert space and let L be a densely defined positive closed selfadjoint linear operator on H satisfying:

A1: $\exists r_1 > 0 : (I + L)^{-r_1}$ is Hilbert-Schmidt on H.

Throughout the present chapter Φ will denote the countably Hilbert nuclear space generated by $(I + L)$ (see Appendix) and Φ' will denote the strong dual of Φ while $(\Phi_r;\langle.,.\rangle)_{r\in\mathbb{R}}$ denotes the associated Hilbert chain.

Let $m \in \Phi'$ and let $Q : \Phi\times\Phi \to \mathbb{R}$ be a strictly positive continuous bilinear map. By the Kernel theorem for nuclear spaces we have

A2: $\exists r_2 \geq 0 \; \exists \theta_2 > 0 \quad \forall \phi,\psi \in \Phi$:

$$|m[\phi]m[\psi] + Q(\phi,\psi)| \leq \theta_2\|\phi\|_{r_2}\|\psi\|_{r_2}.$$

Φ'-valued random variables and stochastic processes are defined in Appendix.

DEFINITION

A Φ'-valued process $W = (W_t)_{t\geq 0}$ (defined on some probability space) is called a Φ'-valued Wiener process with parameters m and Q iff

(i) $\forall\ \phi \in \Phi$: $W_t[\phi]$ is a Gaussian process with mean $tm[\phi]$ and covariance $\mathrm{Cov}(W_t[\phi], W_s[\phi]) = t\ s\ Q(\phi,\phi)$

(ii) $t \to W_t[\phi]$ is continuous with probability one for each $\phi \in \Phi$.

REMARK

If W is a Φ'-valued Wiener process then (i) implies that $W_{t_4} - W_{t_3} \perp\!\!\!\perp W_{t_2} - W_{t_1}$ for any $t_4 \geq t_3 \geq t_2 \geq t_1 \geq 0$;

i.e. a Φ'-valued Wiener process has independent increments.

II.1.1. THEOREM

Let m and Q be as above. Then there exists a probability space $(\Omega, \mathcal{F}, P)$ and a Φ'-valued Wiener process W on $(\Omega, \mathcal{F}, P)$ with parameters m and Q. In fact, if $q \geq r_1 + r_2$ then

$$W \in C([0,\infty), \Phi_{-q}) \quad P\text{-a.s.}.$$

—The theorem was proved by K. Itô [12] for the case $m = 0$ and $\Phi = \mathcal{S}(\mathbb{R})$, whereas V. Perez-Abreu [24] has proved the result for $m = 0$ and any Φ generated in the manner considered here. The necessary alterations of the proof when $m \neq 0$ are straight forward and therefore omitted.

In the sequel we take all random variables and processes to be defined on $(\Omega, \mathcal{F}, P)$ which we assume to be complete. Let η be a Φ'-valued random variable and let

$$\mathcal{G}_t := \{\eta, W_s : 0 \leq s \leq t\} \vee \{P\text{-null sets}\};\ t \geq 0$$

where W is a Φ'-valued Wiener process with parameters m and Q.

Recall that a real stochastic process X is called progressively measurable wrt. $(\mathcal{G}_t)_{t \geq 0}$ iff

(i) X_t is $\mathcal{G}_t$-measurable $\forall\ t \geq 0$

and

(ii) $\forall\ t > 0 : (s,\omega) \to X_s(\omega);\ s \in [0,t]$ is $\mathcal{B}(\mathbb{R})/\mathcal{B}([0,t] \times \mathcal{G}_t$-measurable.

The assumptions on L imply that $L\Phi \subset \Phi$ and that L is continuous on Φ (see proposition III.1.13.). Let L' denote the adjoint of L considered as a continuous linear operator on Φ.

DEFINITION

A Φ'-valued stochastic process $\xi = (\xi_t)_{t \geq 0}$ is a solution to the SDE on Φ' :

(1) $d\xi_t = -L'\xi_t dt + dW_t; \quad \xi_0 = \eta$

iff

(2) $\forall\, \phi \in \Phi : (\xi_t[\phi])_{t\geq 0}$ is progressively measurable wrt. $(\mathcal{G}_t)_{t\geq 0}$.

and

(3) $P\{\xi_t[\phi] = \eta[\phi] + \int_0^t \xi_s[-L[\phi]ds + W_t[\phi],$
$\forall\, \phi \in \Phi\} = 1 \quad \forall\, t \geq 0.$

Moreover, ξ is the unique solution iff for any other Φ'-valued process $(\zeta_t)_{t\geq 0}$ satisfying (2) and (3) we have

$P\{\xi_t = \zeta_t \quad \forall\, t \geq 0\} = 1.$

A1 and selfadjointness of L on H imply the existence of a CONS $\{\phi_j : j \in \mathbb{N}\}$ in H consisting of eigenvectors of L; $L\phi_j = \lambda_j\phi_j$; where $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ with $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$, and where $\phi_j \in \Phi \;\forall\, j \in \mathbb{N}$; see Appendix. Further, $-L$ is dissipative selfadjoint and closed on H and hence $-L$ generates a selfadjoint contraction semigroup $\{T_t : t \geq 0\}$ on H and

$T_t\phi_j = \exp(-\lambda_j t)\phi_j \quad \forall\, j \in \mathbb{N}.$

For each $j \in \mathbb{N}$ let ξ^j denote the unique solution to the real valued SDE

$$d\xi_t^j = -\lambda_j \xi_t^j dt + dW_t[\phi_j]$$

(4)

$$\xi_o^j = \eta[\phi_j],$$

i.e. ξ_t^j is the one-dimensional Ornstein-Uhlenbeck process

(5) $$\xi_t^j = e^{-\lambda_j t}\eta[\phi_j] + \int_o^t e^{-\lambda_j (t-s)} m[\phi_j]ds + \int_o^t e^{-\lambda_j(t-s)} dW_s^j, \text{ where}$$

$$W_s^j := W_s[\phi_j] - sm[\phi_j].$$

<u>II.1.2.</u> <u>THEOREM</u>

Suppose that η satisfies

<u>A3:</u> $\exists r_3 \geq 0 : E\|\eta\|^2_{-r_3} < \infty.$

Then the equation (1) has a unique solution $\xi = (\xi_t)_{t \geq 0}$ given by

(6) $$\xi_t = \sum_{j=1}^{\infty} \xi_t^j \phi_j,$$

the series converging uniformly on [0,T] in the Φ_{-q}-topology (P-a.s.) for any $T > 0$ and any $q \geq (r_1 + r_2) \vee r_3$, where

ξ_t^j is the solution to (4).

Moreover, ξ has the strict Markov property; i.e. $\mathcal{Y}_t$ is conditionally independent of $\sigma\{\xi_s : s \geq t\}$ given $\{\xi_t\}$, and ξ satisfies

$$\xi \in C([0,\infty),\Phi_{-q}) \quad \forall\, q \geq (r_1 + r_2)\, r_3.$$

—The theorem was proved by G. Kallianpur and R. Wolpert in [14]. Their proof for the case where $H = L^2(\mathcal{X},\mathcal{B},\Gamma)$ for a σ-finite measure space $(\mathcal{X},\mathcal{B},\Gamma)$ and where

$$Q(\phi,\psi) = \int_{\mathbb{R}\times\mathcal{X}} a^2\phi(x)\psi(x)\mu(da\,dx)$$

for some σ-finite measure μ on $\mathbb{R}\times\mathcal{X}$, extends without change to any real separable Hilbert space H and any continuous bilinear operator Q on Φ.

If ξ_t is given by (6), then

$$\xi_t[\phi_j] = \xi_t^j \quad (P\text{—a.s.}) \quad \forall\, j \in \mathbb{N},$$

i.e. $\xi_t[\phi_j]$ is a one-dimensional Ornstein-Uhlenbeck process. Therefore, and because of the formal similarity between (1) and a one-dimensional Ornstein-Uhlenbeck equation, we shall call $(\xi_t)_{t\geq 0}$ a Φ'-valued Ornstein-Uhlenbeck process with parameters m,Q and L.

Before proceeding to the investigation of the properties of ξ_t, we need two more results:

Let $N \in \mathbb{N}$ and let P_N denote the orthogonal projection onto $\mathrm{span}\{\phi_j : j \in \{1,\dots,N\}\}$ in Φ_{-q}; where $q \geq (r_1 + r_2) \vee r_3$. Let $(X_t^N)_{t \geq 0} := (P_N \xi_t)_{t \geq 0}$. Then X^N is a Φ_{-q} valued process. Define, for any stochastic processes $Y = (Y_t)_{t \geq 0}$

$$Y^T := (Y_t)_{t \in [0,T]}; \quad \text{where } T > 0.$$

Then $X^{N,T} \in C([0,T],\Phi_{-q})$ (P—a.s.) $\forall\, T > 0$ and we have:

II.1.3. PROPOSITION

$$\forall\, T > 0 : X^{N,T} \underset{N\to\infty}{\Longrightarrow} \xi^T \text{ on } C([0,T],\Phi_{-q}).$$

PROOF:

By theorem II.1.2., for each $T > 0$ we have

$$\sup_{0 \leq t \leq T} \|X_t^{N,T} - \xi_t^T\|_{-q} \underset{N\to\infty}{\longrightarrow} 0 \quad \text{(P—a.s.)},$$

i.e. $X^{N,T}$ converges P—a.s. to ξ^T in the topology of $C([0,T],\Phi_{-q})$. Hence

$$\int_\Omega f(X^{N,T})dP \underset{N\to\infty}{\longrightarrow} \int_\Omega f(\ ^T)dP$$

for any bounded continuous $f : C([0,T],\Phi_{-q}) \to \mathbb{R}$, by the DCT.

Recall from Appendix that if $\phi,\psi \in \Phi$ then

$$\langle\phi,\psi\rangle_r = \sum_{j=1}^{\infty} \langle\phi,\phi_j\rangle_o \langle\psi,\phi_j\rangle_o (1 + \lambda_j)^{2r} \quad \forall\, r \in \mathbb{R}.$$

Note also that, by construction of Φ, $(I + L)^r$ is defined on Φ for any $r \in \mathbb{R}$. By selfadjointness of $(I + L)$ on $\Phi_o = H$ we have for any $\phi,\psi \in \Phi$ and any $r,p \in \mathbb{R}$:

$$< (I + L)^{r-p}\phi,(I + L)^{r-p}\,\psi>_p =$$

$$\sum_{j=1}^{\infty} < (I + L)^{r-p}\,\phi,\phi_j>_o < (I + L)^{r-p}\,\psi,\phi_j>_o (I + \lambda_j)^{2p} =$$

$$\sum_{j=1}^{\infty} <\phi,(I + L)^{r-p}\,\phi_j>_o <\psi,(I + L)^{r-p}\,\phi_j>_o (1 + \lambda_j)^{2p} =$$

$$\sum_{j=1}^{\infty} <\phi,\phi_j>_o <\psi,\phi_j>_o (1 + \lambda_j)^{2r} = <\phi,\psi>_r.$$

II.1.4. THEOREM

For any $r,p \in \mathbb{R}$ there is a unique extension F_r^p of $(I + L)^{r-p}$ to an isometric isomorphism $\Phi_r \to \Phi_p$.

PROOF:

Let $x \in \Phi_r$ and choose $\{\psi_n : n \in \mathbb{N}\} \subset \Phi$ such that $\|x - \psi_n\|_r \to 0$.

Then $(\psi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Φ_r. Now

$$\|\psi_n - \psi_m\|_r^2 = \langle\psi_n - \psi_m, \psi_n - \psi_m\rangle_r =$$

$$\langle (I + L)^{r-p}(\psi_m - \psi_n), (I + L)^{r-p}(\psi_m - \psi_n) \rangle_p,$$

so $((I + L)^{r-p} \psi_n)_{n\in\mathbb{N}}$ is Cauchy in Φ_p.

Let $\hat{x}$ denote its limit in Φ_p. We claim that $\hat{x}$ does not depend on the approximating sequence $\{\psi_n\}_{n\in\mathbb{N}}$. Indeed, let χ_n be another sequence in Φ_r such that $\|x - \chi_n\|_r \xrightarrow[n\to\infty]{} 0$. Let y denote the limit of

$(I + L)^{r-p}\chi_n$ in Φ_p. Then

$$\|\hat{x} - y\|_p \leq \|\hat{x} - (I + L)^{r-p}\psi_n\|_p +$$

$$\|(I + L)^{r-p}(\psi_n - \chi_n)\|_p + \|y - (I + L)^{r-p}\chi_n\|_p$$

Now, $\|\hat{x} - (I + L)^{r-p} \psi_n\|_p$ and $\|y - (I + L)^{r-p}\chi_n\|_p$

both tend to zero as $n \to \infty$ by definition and

$$\|(I + L)^{r-p}(\psi_n - \chi_n)\|_p^2 = \|\psi_n - \chi_n\|_r^2 \underset{n\to 0}{\longrightarrow} 0,$$

since $\psi_n \to x$ and $\chi_n \to x$ in Φ_n as $n \to \infty$.

Hence $\|\tilde{x} - y\|_p = 0$, showing that $\tilde{x}$ is independent of the approximating sequence $\psi_n \to x$ in Φ_r. Therefore, the prescription

$$\Phi_r \ni x \to F_r^p x := \tilde{x} = \lim_{n\to\infty} (I + L)^{r-p} \psi_n \text{ in } \Phi_p$$

defines a (linear) map $F_r^p : \Phi_r \to \Phi_p$. Moreover,

$$\big|\|(I + L)^{r-p} \psi_n\|_p - \|\tilde{x}\|_p\big| \underset{n\to\infty}{\longrightarrow} 0$$

but

$$\|(I + L)^{r-p} \psi_n\|_p = \|\psi_n\|_r \underset{n\to\infty}{\longrightarrow} \|x\|_r$$

so $\|x\|_r = \|\tilde{x}\|_p$ and hence F_r^p is isometric.

Since F_r^p is obviously an extension of $(I + L)^{r-p}$, it only remains to show that F_r^p is surjective:

Let $y \in \Phi_p$. Then $x = F_p^r y \in \Phi_r$ and if $y_n \underset{n\to\infty}{\longrightarrow} y$ in Φ_p,

where $y_n \in \Phi$, we have $x = \lim_{n\to\infty} (I + L)^{p-r} y_n$ in Φ_r.

Further, with $x_n := (I + L)^{p-r} y_n$, we have

$(I + L)^{r-p} x_n = y_n$. Hence $y = F_r^p x$, and $F_r^p = (F_p^r)^{-1}$.

$\Downarrow$

II.2. STATIONARITY AND ABSOLUTE CONTINUITY

In this section we shall show the existence of a unique Gaussian invariant measure for equation (1) and investigate the absolute continuity of the transition measure of the Markov process wrt. the invariant measure.

For convenience we shall assume that $\lambda_1 > 0$. Since $\lambda_1 \le \lambda_2 \le \ldots$ this implies that $\lambda_j > 0 \;\forall\; j$.

We begin by showing that the series

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (\lambda_j + \lambda_k)^{-1} Q(\phi_j,\phi_k) \langle\phi,\phi_j\rangle_o \langle\psi,\phi_k\rangle_o$$

is absolutely convergent for any $\phi,\psi \in \Phi$:

Let $q \ge r_1 + r_2$

$$\sum_{j,k=1}^{\infty} |(\lambda_j + \lambda_k)^{-1} \langle\phi,\phi_j\rangle_o \langle\psi,\phi_k\rangle_o Q(\phi_j,\phi_k)|$$

$$\le \sum_{j,k=1}^{\infty} (2\lambda_1)^{-1} |\langle\phi,\phi_j\rangle_o \langle\psi,\phi_k\rangle_o Q(\phi_j,\phi_k)|$$

$\downarrow$ by A2

$$\leq \sum_{j,k=1}^{\infty} (2\lambda_1)^{-1} |\langle\phi,\phi_j\rangle_o \; \langle\psi,\phi_k\rangle_o| \; \theta_2 \|\phi_j\|_{r_2} \|\phi_k\|_{r_2}$$

$$= \sum_{j,k=1}^{\infty} (2\lambda_1)^{-1} |\langle\phi,\phi_j\rangle_o \; \langle\psi,\phi_k\rangle_o| \; \theta_2 (1 + \lambda_j)^{r_2} (1 + \lambda_k)^{r_2}$$

$$= \sum_{j,k=1}^{\infty} \theta_2 (2\lambda_1)^{-1} |\langle\phi,\phi_j\rangle_o (1 + \lambda_j)^q \; || \langle\psi,\phi_k\rangle_o (1 + \lambda_k)^q |$$

$$(1 + \lambda_j)^{r_2 - q} (1 + \lambda_k)^{r_2 - q}$$

(by Cauchy-Schwartz and choice of q)

$$\leq \theta_2 (2\lambda_1)^{-1} (\|\phi\|_q^2 \|\psi\|_q^2)^{1/2} \sum_{j=1}^{\infty} (1 + \lambda_j)^{-2r_1}$$

$$= \theta_1 \theta_2 (2\lambda_1)^{-1} \|\phi\|_q \|\psi\|_q < \infty, \text{ since}$$

$$\theta_1 := \sum_{j=1}^{\infty} (1 + \lambda_j)^{-2r_1} < \infty \text{ by A1.}$$

hence

$$\left| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (\lambda_j + \lambda_k)^{-1} \langle\phi,\phi_j\rangle_o \; \langle\psi,\phi_k\rangle_o \; Q(\phi_j,\phi_k) \right|$$

$$\leq \theta_1 \theta_2 (2\lambda_1)^{-1} \|\phi\|_q \|\psi\|_q \quad \forall \; \phi,\psi \in \Phi$$

and since Φ is dense in Φ_q a continuous bilinear map B may be defined on Φ_q by

$$B(\phi,\psi) := \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} (\lambda_j + \lambda_k)^{-1} \langle\phi,\phi_j\rangle_o \langle\psi,\phi_k\rangle_o Q(\phi_j,\phi_k);$$

$$\phi,\psi \in \Phi_q.$$

Define a continuous linear map $S : \Phi_{-q} \to \Phi_{-q}$ by requiring

$$\langle Su,v\rangle_{-q} = B(F^q_{-q}u, F^q_{-q}v), \quad \forall\, u,v \in \Phi_{-q}.$$

Then S is positive, selfadjoint and nuclear with

$$Tr(S) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} (\lambda_j + \lambda_k)^{-1} Q(\phi_j,\phi_k)(1 + \lambda_j)^{-q}(1 + \lambda_k)^{-q}.$$

Define a continuous linear map $\Lambda : \Phi \to \Phi$ by

$$\Lambda\phi = \sum_{j=1}^{\infty} \lambda_j^{-1}\langle\phi,\phi_j\rangle_o\, \phi_j.$$

Then, for any $r \geq 0$,

$$\|\Lambda\phi\|_r \leq \lambda_1^{-1}\|\phi\|_r$$

and hence Λ extends to a continuous linear map: $\Phi_r \to \Phi_r$ for every $r \geq 0$.

Now, the mapping

$$\Phi_{-q} \ni y \to m[\Lambda F^q_{-q}y]$$

defines a continuous linear functional on Φ_{-q} and therefore there is $\bar{m} \in \Phi_{-q}$ such that

$$m[\Lambda F^q_{-q} y] = \langle \bar{m}, y \rangle_{-q} \quad \forall\, y \in \Phi_{-q}.$$

Define, for $\phi \in \Phi_q$,

$$C_q(\phi) = \exp(im[\Lambda\phi] - 1/2B(\phi,\phi)).$$

Since (by theorem II.1.4) any $\phi \in \Phi_q$ has the form $\phi = F^q_{-q} y$ for a unique $y \in \Phi_{-q}$ we have

$$C_q(y) = \exp(im[\Lambda F^q_{-q} y] - 1/2B(F^q_{-q} y, F^q_{-q} y))$$

$$= \exp(i\langle \bar{m}, y \rangle_{-q} - 1/2\langle Sy, y \rangle_{-q});$$

i.e. $C_q(y)$ is the characteristic functional of the Gaussian measure on Φ_{-q} with mean functional $\bar{m}$ and covariance operator S. We shall denote this measure by $\nu = N_{-q}(\bar{m}, S)$.

-In the sequel, whenever we talk about initial conditions for SDE's on Φ' we shall tacitly assume that they satisfy A3.

DEFINITION:

A Borel measure μ on Φ' is called an invariant measure for the SDE on Φ'

$$(7) \quad \begin{cases} d\xi_t = -L'\xi_t dt + dW_t \\ \xi_o = \eta \end{cases}$$

iff, whenever η has distribution μ and $\eta \perp\!\!\!\perp \{W_s : s \geq 0\}$,

$P(\ _t \in A) = \mu(A) \quad \forall A \in \mathcal{B}(\Phi') \quad \forall t > 0.$

-Note that since Φ' is the <u>strict</u> inductive limit of Φ_{-r}; $r \geq 0$, $\mathcal{B}(\Phi')$ relativized to Φ_{-r} is equal to $\mathcal{B}(\Phi_{-r})$. Therefore, any Borel measure μ on Φ_{-r} can be extended to a Borel measure on Φ' by identifying μ with μ^* defined by

$\mu^*(A) = \mu(A \cap \Phi_{-r})$; $A \in \mathcal{B}(\Phi')$.

Henceforth we shall regard measures on Φ_{-r} as extended in this way.

<u>THEOREM II.1.1.</u>

Let $q = r_1 + r_2$. Then $\nu = N_{-q}(\bar{m}, S)$ is an invariant measure for equation (7). Moreover, if μ is any other invariant measure then

$\mu(A) = \nu(A) \quad \forall A \in \mathcal{B}(\Phi').$

<u>PROOF:</u>

Let η be independent of $\{W_s : s \geq 0\}$ and have distribution $N_{-q}(\bar{m},S)$.

Then $E\|\eta\|_{-q}^2 < \infty$, so (7) has a unique solution by theorem II.1.2 given by

$$\xi_t = \sum_{j=1}^{\infty} \xi_t^j \phi_j.$$

Let $N \in \mathbb{N}$. The $\mathbb{R}^N$-valued process $Y_t^N = (\xi_t^1,\ldots,\xi_t^N)'$ satisfies

$$(8) \qquad \begin{aligned} dY_t^N &= L_N Y_t dt + dZ_t \\ Y_o &= (\eta[\phi_1],\ldots,\eta[\phi_N])' \end{aligned}$$

where

$(L_N)_{ij} = \lambda_j \delta_{ij}$; $i,j = 1,\ldots,N$ and

$Z_t = (W_t[\phi_1],\ldots,W_t[\phi_N])$

i.e. Y_t^N is given by

$$Y_t^N = S_t^N Y_o + \int_o^t S_{t-s}^N dZ_t$$

where $(S_t^N)_{ij} = e^{-\lambda_j t}\delta_{ij}$; $j,i = 1,\ldots,N$.

Hence Y_t^N is a Gaussian process, and a computation will verify that

$$EY_t^N = (m[\wedge\phi_1],\ldots,m[\wedge\phi_N])' \quad \forall\, t \geq 0 \text{ and}$$

$$\mathrm{Var}(Y_t^N)_{ij} = B(\phi_i,\phi_j);\ i,j = 1,\ldots,N \quad \forall\, t \geq 0$$

Let $F_N : \Phi_{-q} \to \mathbb{R}$ denote the map given by

$$F_N(x) = (x_1,\ldots,x_N)'; \text{ where}$$

$$x = \sum_{j=1}^{\infty} x_j\phi_j; \quad \text{with}$$

$$\sum_{j=1}^{\infty} x_j(1 + \lambda_j)^{2q} < \infty.$$

Then

$$X_t^N = F_N^{-1}(Y_t^N),$$

(recall from page 13 that $X_t^N = \sum_{j=1}^{N} {}_t^j\phi_j$.

Fix $t > 0$. Let C_t^N denote the characteristic function of Y_t^N; i.e.

$$C_t^N(y_1,\ldots,y_N) = \exp(i\sum_{j=1}^{N} y_j m[\wedge\phi_j]$$

$$-\frac{1}{2}\sum_{j,k=1}^{N} y_j y_k B(\phi_j,\phi_k))$$

Let $\phi \in \Phi_q$; $\phi = \sum_{j=1}^{\infty} \langle\phi,\phi_j\rangle_o \phi_j$ (converging in Φ_q).

Then the characteristic functional of X_t^N (evaluated at ϕ) is

$$K_t^N(\phi) = C_t^N(\langle\phi,\phi_1\rangle_o,\ldots,\langle\phi,\phi_N\rangle_o)$$

$$= \exp(i\sum_{j=1}^{N} \langle\phi,\phi_j\rangle_o m[\wedge\phi_j]$$

$$-\frac{1}{2}\sum_{k;j=1}^{N} \langle\phi,\phi_j\rangle_o \langle\phi,\phi_k\rangle_o B(\phi_j,\phi_k))$$

$$= \exp(im[\wedge\sum_{j=1}^{N} \langle\phi,\phi_j\rangle_o \phi_j]$$

$$-\frac{1}{2}B(\sum_{j=1}^{N} \langle\phi,\phi_j\rangle_o \phi_j, \sum_{k=1}^{N} \langle\phi,\phi_k\rangle_o \phi_k))$$

Now, $\sum_{j=1}^{N} \langle\phi,\phi_j\rangle_o \phi_j \xrightarrow[N\to\infty]{} \phi$ in Φ_q and since B and $m\Lambda$ are continuous on Φ_q we get

$$\lim_{N\to\infty} K_t^N(\phi) = K_t(\phi) = \exp(im[\Lambda\phi] - \frac{1}{2}B(\phi,\phi))$$

i.e. K_t is the characteristic functional of the measure $\nu = N_{-q}(\bar{m},s)$ (c.f. page 20).

Now, lemma II.1.3 implies that

$$X_t^N \underset{N\to\infty}{\Longrightarrow} \xi_t \text{ for each } t \geq 0.$$

Hence K_t^N must converge to the characteristic functional of ξ_t, i.e.

$$K_t(\phi) = E\exp(i\xi_t[\phi])$$

But K_t was just shown to be equal to the characteristic functional of ν.

Hence

$$P(\xi_t \in A) = \nu(A) \quad \forall A \in \mathcal{B}(\Phi'),$$

concluding the existence part.

Next let μ be an invariant measure for equation (7). Let η have distribution μ and be independent of $\{W_s : s \geq 0\}$. By theorem II.1.2 there is $q \geq r_1 + r_2$ such that the solution ξ_t to (7) satisfies $\xi_t \in \Phi_{-q}$ $\forall\ t \geq 0$. Let P_N denote the orthogonal projection onto span$\{\phi_j : j = 1,\ldots,N\}$ in Φ_{-q}. Let F_N be as in the first part of the proof and let Y_t^N denote the unique solution to (8), where η now has distribution μ. Then, for any $B \in \mathcal{B}(\mathbb{R}^N)$, we have

$$\mu \circ P_N^{-1} \circ F_N^{-1}(B) = P(F_N P_N\ _t \in B) \quad \forall\ t \geq 0$$

$$= P(Y_t^N \in B) \quad \forall\ t \geq 0.$$

Hence

$\mu \circ P_N^{-1} \circ F_N^{-1}$ is an invariant measure for the ordinary SDE (8). But the unique invariant measure for this equation is the Gaussian measure ν_N on $\mathbb{R}^N$ with mean $(m[\Lambda\phi_1],\ldots,m[\Lambda\phi_N])'$ and covariance matrix $(\Sigma)_{ij} = B(\phi_i,\phi_j)$, $i,j=1,\ldots,N$.

Hence

$$\mu \circ P_N^{-1} \circ F_N^{-1}(B) = \nu_N(B) \forall\ B \in \mathcal{B}(\mathbb{R}^N).$$

But $\nu_N = \nu \circ P_N^{-1} \circ F_N^{-1}$ for each $N \in \mathbb{N}$.

Since N was arbitrary, we get

$$\mu \circ P_N^{-1} \circ F_N^{-1}(B) = P_N^{-1} \circ F_N^{-1}(B) \quad B \in \mathcal{B}(\mathbb{R}^N) \quad \forall\ N \in \mathbb{N}.$$

But

$$\sigma\{A \subset \Phi_{-q} : \exists N \in \mathbb{N}\ \exists B \in \mathcal{B}(\mathbb{R}^N) : A = P_N^{-1} \circ F_N^{-1}(B)\}$$

$$= \mathcal{B}(\Phi_{-q}), \text{ so } \mu(A) = \nu(A) \quad \forall A \in \mathcal{B}(\Phi_{-q}).$$

But $\mu(C) = \mu(C \cap \Phi_{-q}) \quad \forall C \in \mathcal{B}(\Phi')$

because, by invariance property, $\mu(C) = P(\xi_t \in C)$ for any $t \geq 0$ and $\xi_t \in \Phi_{-q}$ P-a.s. $\forall$ t.

Hence, for any $C \in \mathcal{B}(\Phi')$

$$\mu(C) = \mu(C \cap \Phi_{-q}) = \nu(C \cap \Phi_{-q}) =$$

$$(C \cap \Phi_{-(r_1+r_2)}) = \nu(C)$$

(note that $q \geq r_1 + r_2$ and that $C \cap \Phi_{-q} \in \mathcal{B}(\Phi')$, so that

$$(C \cap \Phi_{-q}) = \nu(C \cap \Phi_{-(r_1+r_2)}) = \nu(C),$$

by the convention of identifying ν with its extension to $\mathcal{B}(\Phi')$).

We shall not give a general and thorough discussion of stationary solutions to Φ'-valued SDE's. The following considerations will suffice for our purpose:

For an ordinary Ornstein–Uhlenbeck SDE, starting at an initial condition whose distribution is equal to the invariant measure for that equation, produces a stationary solution in the sense of K. Itô, [13]. This stationary solution is defined for all $t \in \mathbb{R}$ and is a wide sense stationary process X_t which has distribution equal to the invariant measure for every $t \in \mathbb{R}$.

We shall now see that also the Φ'–valued Ornstein-Uhlenbeck process can be extended to a Φ'–valued process ξ_t defined for all $t \in \mathbb{R}$, which is wide sense stationary and whose distribution is equal to the invariant measure (for all $t \in \mathbb{R}$).

For each $N \in \mathbb{N}$ let $Y_t^N = (\xi_t^j)_{j=1}^N$ denote the stationary solution to the SDE

$$dY_t^N = - L_N Y_t^N dt + dZ_t;\ t \in \mathbb{R};\ \text{i.e.}$$

$$\xi_t^j = \int_{-\infty}^{t} e^{-\lambda_j (t-s)} dW_s[\phi_j];\ j = 1,\dots,N$$

notice that if $t > 0$ then

$$\xi_t^j = e^{-\lambda_j t} \xi_o^j + \int_o^t e^{-\lambda_j (t-s)} dW_s[\phi_j],$$

where

$$\xi_o^j = \int_{-\infty}^{o} e^{\lambda_j s} dW_s[\phi_j] \text{ and}$$

the joint distribution of $(\xi_o^1,\dots,\xi_o^N)$ is $N((m[\phi_1],\dots,m[\phi_N]),\{B(\phi_j,\phi_k)\})$.

Let $\eta := \sum_{j=1}^{\infty} \xi_o^j \phi_j$. Then

$$E\sum_{j=1}^{\infty} (\xi_o^j)^2 \|\phi_j\|_{-q}^2 =$$

$$\sum_{j=1}^{\infty} (B(\phi_j,\phi_j) + (m[\Lambda\phi_j])^2)(1 + \lambda_j)^{-2q} \le$$

$$\sum_{j=1}^{\infty} (\lambda_1^{-2}\vee\Lambda_1^{-1})(Q(\phi_j,\phi_j) + (m[\phi_j])^2)(1 + \lambda_j)^{-2q} \le$$

(by A2)

$$\sum_{j=1}^{\infty} (\lambda_1^{-2}\vee\lambda_1^{-1})\theta_2\|\phi_j\|_{r_2}^2(1 + \lambda_j)^{-2q} =$$

$$\sum_{j=1}^{\infty} (\lambda_1^{-2}\vee\lambda_1^{-1})\ \theta_2(1 + \lambda_j)^{-2r_1} < \infty, \text{ by A1}$$

and so

$$\sum_{j=1}^{\infty} (\xi_o^j)^2 \|\phi\|_{-q}^2 < \infty \quad \text{P-a.s.}$$

Hence $\eta \in \Phi_{-q}$ (P-a.s.) and $E\|\eta\|_{-q}^2 < \infty$.

It now follows from theorem II.1.2 that

$\xi_t := \sum_{j=1}^{\infty} \xi_t^j \phi_j$ $(t \geq 0)$ is the unique solution to the

Φ'-valued SDE

$$\left.\begin{aligned} d\xi_t &= -L'\xi_t dt + dW_t \\ \xi_o &= \eta \end{aligned}\right. \quad ; \ t \geq 0$$

Moreover, the characteristic functional of η is

$$C(\phi) = \lim_{N\to\infty} \exp\left[i\sum_{j=1}^{N} [\wedge\phi_j]\langle\phi,\phi_j\rangle_o - \frac{1}{2}\sum_{j=1}^{N} \langle\phi,\phi_j\rangle_o\langle\phi,\phi_k\rangle_o B(\phi_j,\phi_k) \right]$$

$$= \exp(im[\wedge\phi_j] - \frac{1}{2}B(\phi,\phi))$$

where $\Phi_q \ni \phi = \sum_{j=1}^{\infty} \langle\phi,\phi_j\rangle_o \phi_j$

i.e. η has distribution $\nu = N_{-q}(\bar{m},S)$.

Since $\eta[\phi_j] = \zeta_0^j = \int_{-\infty}^{0} e^{\lambda_j s} dW_s[\phi_j] \quad \forall\, j \in \mathbb{N}$

η is obviously independent of $\{W_s : s \geq 0\}$.

Now, let $t \in \mathbb{R}$. Then

$$E\sum_{j=1}^{\infty} (\zeta_t^j)^2 \|\phi_j\|_{-q}^2 =$$

$$\sum_{j=1}^{\infty} E(\eta_t^j)^2 \|\phi_j\|_{-q}^2 =$$

$$\sum_{j=1}^{\infty} (e^{-2\lambda_j t}\int_{-\infty}^{t} e^{2\lambda_j s} Q(\phi_j,\phi_j)ds$$

$$+ (\int_{-\infty}^{t} e^{-\lambda_j(t-s)} ds\ m[\phi_j])^2)(1 + \lambda_j)^{-2q} =$$

$$\sum_{j=1}^{\infty} (Q(\phi_j,\phi_j)\frac{1}{2\lambda_j} + (\frac{1}{\lambda_j} m[\phi_j])^2)(1 + \lambda_j)^{-2q}$$

$$\leq \sum_{j=1}^{\infty} (\lambda_1^{-1} \vee \lambda_1^{-2})\theta_2 \|\phi\|_{r_2}^2 (1 + \lambda_j)^{-2q}$$

$$= \sum_{j=1}^{\infty} (\lambda_1^{-1} \vee \lambda_1^{-2})\theta_2 (1 + \lambda_j)^{-2r_1} < \infty.$$

Hence there is a $\Omega_t \in \mathcal{F}$ with $P(\Omega_t) = 1$

$$\sum_{j=1}^{\infty} (\tilde{\xi}_t^j(\omega))^2 \|\phi_j\|_{-q}^2 < \infty \quad \forall \omega \in \Omega_t.$$

Define

$$\tilde{\xi}_t(\omega) = \begin{cases} \sum_{j=1}^{\infty} \tilde{\xi}_t^j(\omega)\phi_j \text{ if } \omega \in \Omega_t \\ 0 \text{ if } \omega \notin \Omega_t. \end{cases}$$

Then $(\tilde{\xi}_t)_{t\in\mathbb{R}}$ is a Φ_{-q}-valued process and

$$\tilde{\xi}_t = \xi_t \quad \text{(P-a.s.)} \quad \forall t \geq 0;$$

where ξ_t is the unique solution to

$$d\xi_t = -L'\xi_t dt + dW_t$$

$$\xi_o = \eta;$$

with $\eta \sim N_{-q}(\bar{m},S)$ and $\eta \perp\!\!\!\perp \{W_s : s \geq 0\}$.

DEFINITION

A Φ'-valued process $X = (X_t)_{t\in\mathbb{R}}$ is called (wide sense) stationary iff

i) $\forall \phi \in \Phi : EX_t[\phi]$ does not depend on t.

ii) $\forall\ \phi,\psi \in \Phi$: $\mathrm{Cov}(X_t[\phi],X_s[\psi])$ is a function of only $(t-s,\phi,\psi)$, $t \geq s \in \mathbb{R}$.

II.2.2. THEOREM

$\xi = (\xi_t)_{t\in\mathbb{R}}$ is a wide sense stationary process. Moreover, for each $t \in \mathbb{R}$ the distribution of ξ_t is equal to the invariant measure for equation (7).

PROOF:

Let $\phi,\psi \in \Phi$. Then,

$$E\xi_t[\phi] = \sum_{j=1}^{\infty} E\xi_t^j \langle\phi,\phi_j\rangle_o$$

$$= \sum_{j=1}^{\infty} m[\phi_j]\lambda_j^{-1}\langle\phi,\phi_j\rangle_o$$

$$= m[\Lambda\phi]$$

Next,

$$\mathrm{Cov}(\xi_t[\phi],\xi_s[\psi] =$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle\phi,\phi_j\rangle_o\langle\phi,\phi_k\rangle_o \mathrm{Cov}(\xi_t^j,\xi_s^k)$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle\phi,\phi_j\rangle_o \langle\phi,\phi_k\rangle_o \; e^{-\lambda_j(t-t\ s)} e^{-\lambda_k(s-t\ s)} B(\phi_j,\phi_k)$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle\phi,\phi_j\rangle_o \langle\phi,\phi_k\rangle_o B(\phi_j,\phi_k) h_{jk}(t-s)$$

$$\text{where } h_{jk}(u) := \begin{cases} e^{-\lambda_j|u|} & \text{if } u \geq 0 \\ e^{-\lambda_k|u|} & \text{if } u < 0. \end{cases}$$

Since the series

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle\phi,\phi_j\rangle_o \langle\phi,\phi_k\rangle_o B(\phi_j,\phi_k)$$

was shown earlier to be absolutely convergent, this concludes the proof of the wide sense stationarity of ξ_t.

By construction, for each $t \in \mathbb{R}$ the joint distribution of $(\xi_t^1,\ldots\xi_t^N)$ is Gaussian with mean $(m[\Lambda\phi_1],\ldots,m[\Lambda\phi_N])$ and covariance matrix $(\Sigma)_{ij} = B(\phi_i,\phi_j)$. Moreover, by definition of ξ_t we have

$$\xi_t = \lim_{N\to\infty} \sum_{j=1}^{N} \xi_t^j \phi_j \quad (\text{in } \Phi_{-q}) \quad P\text{-a.s.}$$

Hence the characteristic functional for ξ_t is the limit as

$N\to\infty$ of the characteristic functional C_N of $\sum_{j=1}^{N} \xi_t^j \phi_j$. But

$$C_N(\phi) = \exp\Big[\, i \sum_{j=1}^{N} m[\Lambda\phi_j]\langle\phi,\phi_j\rangle_o -$$

$$1/2 \sum_{j=1}^{N} \langle\phi,\phi_j\rangle_o \langle\phi,\phi_k\rangle_o B(\phi_j,\phi_k) \Big]$$

Hence

$$E\exp(i \ {}_t[\phi]) = \lim_{N\to\infty} C_N(\phi)$$

$$= \exp(im[\Lambda\phi] - 1/2B(\phi,\phi)).$$

Hence $P(\xi_t \in A) = \nu(A) \quad \forall\ A \in \Phi_{-q}$ where $\nu = N_{-q}(\bar{m},S)$ is the invariant measure for equation (7).

II.2.3. PROPOSITION

When L satisfies A1 and $\{T_t : t \geq 0\}$ denotes the selfadjoint contraction semigroup on H generated by L then

(9) $\quad T_t\Phi_r \subset \Phi_r \quad \forall\ r \geq 0 \quad \forall\ t \geq 0$

(10) $\quad T_t|_{\Phi_r}$ is nuclear $\quad \forall\ r \geq 0 \quad \forall\ t > 0.$

PROOF:

Fix $r \geq 0$. Let $\phi \in \Phi_r$. Since $\Phi_r \quad \Phi_o = H$ we have, for any $t \geq 0$

$$T_t\phi = \sum_{j=1}^{\infty} e^{-\lambda_j t} \langle \phi, \phi_j \rangle_o \, \phi_j, \text{ so}$$

$$\|T_t\phi\|_r^2 = \sum_{j=1}^{\infty} e^{-2\lambda_j t} \langle \phi, \phi_j \rangle_o^2 \, (1 + \lambda_j)^{2r}$$

$$\leq \|\phi\|_r^2,$$

and since Φ is dense in Φ_r this proves (9) and also shows that $T_t|_{\Phi_r}$ is $\|\cdot\|_r$-continuous. Hence we only need to show that T_t has finite trace for each $t > 0$:

By A1,

$$\sum_{j=1}^{\infty} (1 + \lambda_j)^{-2r_1} < \infty.$$

For $t > 0$ fixed, $(1 + \lambda_j)^{2r_1} e^{-\lambda_j t} \to 0$ as $j \to \infty$.

Hence $\displaystyle\sum_{j=1}^{\infty} e^{-\lambda_j t} < \infty.$

Since $\left(e^{-\lambda_j t}, \dfrac{\phi_j}{\|\phi_j\|_r}\right)$ is the eigensystem for $T_t|_{\Phi_r}$,

$$\mathrm{Trace}(T_t|_{\Phi_r}) = \sum_{j=1}^{\infty} e^{-\lambda_j t} < \infty.$$

Next, we shall give necessary and sufficient conditions that the transition measure of the Markov process $\xi = (\xi_t)_{t \geq 0}$ be equivalent to the invariant measure $\nu = N_{-q}(\bar{m}, S)$.

Let $P(t|\eta)$ denote the transition measure of the Markov process ξ. For any η in Φ_{-q} $P(t|\eta)$ is a Gaussian measure on Φ_{-q}.

II.2.4. THEOREM

Suppose that $\lambda_1 > 0$. Let η be a Φ_{-q}-valued random variable such that $\eta \in \mathrm{Range}(S)$.

Then, for any $t > 0$, $P(t|\eta)$ and ν are equivalent on Φ_{-q} iff

(11) $\bar{m} \in \mathrm{Range}(S^{\frac{1}{2}})$ and

(12) $T'_t(\mathrm{Range}(S^{\frac{1}{2}}))$ $\mathrm{Range}(S^{\frac{1}{2}})$.

REMARK

By proposition II.2.3., $T_t\Phi_r \subset \Phi_r$ $\forall$ $r \geq 0$ and we saw that $T_t|_{\Phi_r}$ is $\|\cdot\|_r$-continuous. Hence $T_t\Phi \subset \Phi$ and T_t is continuous on Φ. T'_t denotes the adjoint of T_t considered as a continuous linear operator on Φ. It follows from proposition II.2.2. that $T_t'\Phi_{-r} \subset \Phi_{-r}$ $\forall$ r $\geq$ 0 and that $T_t'|\Phi_{-r}$ is nuclear $\forall$ r $\geq$ 0.

PROOF OF THEOREM II.2.4.:

Let t > 0. It is easily checked that $P(t|\eta)$ is a Gaussian measure on Φ_{-q} with mean functional $T_t'(\eta - \bar{m}) + \bar{m}$ and covariance operator $\Sigma_t = S - T_t'ST_t'$. Hence $P(t|\eta)$ and are either equivalent or orthogonal. By the Feldman-Hajek theorem (see H.H. Kuo [11] theorem 3.4 page 125) they are equivalent iff

(13) $\quad T_t'(\eta - \bar{m}) \in \text{Range}(S^{1/2})$

(14) $\quad \mathcal{B}_t = S^{1/2}(I - B_t)S^{1/2}$; where

(15) $\quad B_t : \text{Range}(S^{1/2}) \rightarrow \Phi_{-q}$ is continuous and $I - B_t$ is positive definite

(16) $\quad B_t$ is Hilbert-Schmidt.

Sufficiency of (11) and (12):

Since $\eta \in R(S)$ and $R(S^{1/2}) \supset R(S)$, (11) and (12) imply (13).

Now, $\Sigma_t = S - T_t'ST_t'$ and since

$T'_t(\text{Range}(S^{1/2}) \subset \text{Range}(S^{1/2})$, we have

$$S - T'_t S T'_t = S^{1/2}(I - S^{-1/2}T'_t S T'_t S^{-1/2})S^{1/2}.$$

Define $B_t = S^{-1/2}T'_t S T'_t S^{-1/2}$. Then B_t is well defined on $R(S^{1/2})$ and $I - B_t$ is non-negative definite (because Σ_t and $S^{1/2}$ are positive definite) and

$$B_t = E_t^* E_t, \text{ where}$$

$E_t := S^{1/2}T'_t S^{-1/2}$, and E_t^* denotes the Hilbert-space adjoint of E_t on Φ_{-q}.

Let $\{e_n : n \in \mathbb{N}\}$ be a CONS in Φ_{-q} consisting of eigenvectors of S. Then

$$\langle E_t e_n, e_n\rangle_{-q} = \langle T'_t e_n, e_n\rangle_{-q}, \text{ so}$$

$$(17) \qquad \sum_{n=1}^{\infty} \langle E_t e_n, e_n\rangle_{-q} = \sum_{n=1}^{\infty} \langle T'_t e_n, e_n\rangle_{-q}$$

$$= \text{Trace}(T'_t|_{\Phi_{-q}}) < \infty,$$

since $T'_t|_{\Phi_{-q}}$ is nuclear. Moreover,

$$E_t S^{1/2}\phi_j = e^{-\lambda_j t} S^{1/2}\phi_j \quad \forall\, j \in \mathbb{N};$$

i.e. $e^{-\lambda_j t}$ is an eigenvalue for E_t for each $j \in \mathbb{N}$.

Further, since

$$\{\frac{\phi_j}{\|\phi_j\|_{-q}};\ j \in \mathbb{N}\}$$

is a CONS in Φ_{-q}, span$\{S^{1/2}\phi_j : j \in \mathbb{N}\}$ is dense in $R(S^{1/2})$. Hence there is a complete orthonormal system $\{b_j : j \in \mathbb{N}\} \subset$ span$\{S^{1/2}\phi_j : j \in \mathbb{N}\}$ for $R(S^{1/2})$ such that

$$E_t b_j = e^{-\lambda_j t} b_j \quad \forall\ j \in \mathbb{N}.$$

But then $\sup_{x \in U} \|E_t x\|_{-q} < 1$, where

$U = \{x \in R(S^{1/2}) : \|x\|_{-q} \leq 1\}$ and hence

$E_t : R(S^{1/2}) \to \Phi_{-q}$ is a contraction, in particular continuous. Since $B_t = E_t^* E_t$ and $I - B_t$ has already been shown to be non-negative definite it follows that $I - B_t$ is positive definite.

By (17) E_t has finite trace and thus E_t is nuclear. Hence $B_t = E_t^* E_t$ is nuclear, in particular Hilbert-Schmidt, and continuous : $R(S^{\frac{1}{2}}) \to \Phi_{-q}$. Hence (15) and (16) hold and (14) is immediate from the definition of B_t. Thus (11) and (12) are sufficient for equivalence of $P(t|\eta)$ and .

<u>Necessity of (11) and (12):</u>

If $P(t|\eta)$ and ν are equivalent then (13) through (16) hold. (14) gives

$$S - T_t' S T_t' = S^{1/2}(I - B_t)S^{1/2}; \text{ i.e.}$$

$$(18) \qquad T_t' S T_t' = S^{1/2} B_t S^{1/2}.$$

Since $T_t' S^{1/2}$ and $S^{1/2}$ are positive definite, B_t is positive definite. Hence we may write $B_t = D_t^* D_t$ for some positive definite $D_t : R(S^{1/2}) \to \Phi_{-q}$. But then (since T_t is easily seen to be selfadjoint on Φ_q and hence T_t' is selfadjoint on Φ_{-q}) (18) gives:

$$(T_t' S^{1/2})(T_t' S^{1/2})^* = (S^{1/2} D_t^*) I (S^{1/2} D_t^*)^*$$

and consequently

$$R(T_t' S^{1/2}) = R(S^{1/2} D_t^*)$$

(see e.g. C.R. Baker [1], Corollary 1, page RR2) which implies that $T_t' R(S^{1/2}) \subset R(S^{1/2})$, i.e. (12) holds. But since $\eta \in R(S)$ and $R(S) \subset R(S^{\frac{1}{2}})$, (11) now follows from (14) and (15).

$\not\downarrow$

In the general case the formula for the Radon-Nikodym derivative of $P(t|\eta)$ wrt. ν is impractical, but when $Q(\phi_j, \phi_k) = 0$ whenever $j \neq k$ the coordinate processes

$\xi_t[\phi_j]$ are independent and S and T_t' have the same eigenvectors. In this case a very handy expression for the Radon-Nikodym derivative is available. In addition the case $Q(\phi_j,\phi_k) = 0$ when $j \neq k$ is of interest in the context of [23] and [29].

II.2.5. COROLLARY

Suppose that $\lambda_1 > 0$, $Q(\phi_j,\phi_k) = 0$ if $j \neq k$ and that $m = 0$. If T_t satisfies (12) and $\eta \in R(S) \subset \Phi_{-q}$ then

$$\frac{dP(t|\eta)}{d\nu}(y) = \prod_{j=1}^{\infty}(1 - e^{-2\lambda_j t})^{-1/2}$$

$$\exp\left[-2\lambda_j\sigma_j^{-2}(1 - e^{-2\lambda_j t})^{-1}(e^{-2\lambda_j t}(\eta_j^2 + y_j^2) - 2e^{-\lambda_j t}\eta_j y_j)\right]; \text{ where}$$

$\sigma_j^2 = Q(\phi_j,\phi_j)$ and

η and $y \in R(S)$ are given by

$$\eta = \sum_{j=1}^{\infty} \eta_j\phi_j \text{ and } y = \sum_{j=1}^{\infty} y_j\phi_j,$$

both converging in Φ_{-q}.

—The proof is a straight forward application of theorem 3.3 in Kuo [18], theorem 16.2 page 83 in Skorohod [25] and the formula

$$\frac{dP(t|\eta)}{d\nu} = \frac{dP(t|\eta)}{dP(t|0)} \frac{dP(t|0)}{d\nu}$$

We shall conclude this section by stating a simple sufficient condition for equivalence for the case $Q(\phi_j,\phi_k) = \delta_{jk}\sigma_j^2$:

II.2.6. PROPOSITION

Suppose that $\lambda_1 > 0$ and let $\eta \in R(S)$. If Q has the form

$$Q(\phi_j,\phi_k) = \delta_{jk}\sigma_j^2, \quad \text{for some} \quad \sigma_j^2 \leq \theta_2(1 + \lambda_j)^{r_2}$$

then (11) and (12) are satisfied if

(a) $\exists r_4 > 0 \; \exists N_o \in \mathbb{N} \; \exists C > 0 :$

$$_j^2 \geq C(1 + \lambda_j)^{-r_4} \quad \forall \; j \geq N_o$$

and

(b) $$\sum_{j=1}^{\infty} \lambda_j^{-1} \; _j^{-2}(m[\phi_j])^2 < \infty.$$

REMARK

In [23] Miyahara considers the following set-up:

Let $H = L^2([0,\pi])$ and let $\hat{w} = \sqrt{-\Delta}$; Δ being the Laplace operator with Neumann boundary conditions at 0 and π. Then the eigensystem of $\hat{w}$ is $\{(\phi_j, j) : j = 0,1,2,\dots\}$ where

$$\phi_j(x) = \begin{cases} \pi^{-1/2} & \text{if } j = 0 \\ \dfrac{2}{\pi} \operatorname{Cos} jx & \text{if } j \geq 1. \end{cases}$$

Let $\tilde{H} = \{h \in H : \langle h, \phi_o \rangle_H = 0\}$. Then $\hat{w}$ is strictly positive on $\tilde{H}$ and Miyahara considers the countably Hilbert nuclear space

$$\Phi = \phi \in \tilde{H} : \|\hat{w}^{\alpha} \phi\|_H < \infty \quad \forall \alpha \in \mathbb{R}\}.$$

From a cylindrical Brownian motion on H Miyahara then constructs a Φ'-valued Wiener process B_t with parameters $m = 0$ and $Q(\phi,\psi) = \langle \phi,\psi \rangle_H$; $\phi,\psi \in \Phi$, and proceeds to study the SDE on Φ':

$$dX_t = -\hat{w} X_t dt + dB_t.$$

He shows that there is a unique invariant measure for this equation, and, given any initial condition $\eta \in \Phi_{-\frac{1}{2}}$, the transition probability measure of X_t given η is always equivalent to the invariant measure. Since $m = 0$ and $Q(\phi,\psi) = \langle I\phi,\psi \rangle_H$ in Miyahara's case, (a) and (b) of proposition II.2.6. are satisfied and thus explain why no extra assumptions are needed to ensure the equivalence in

Miyahara's case. Moreover, $\lambda_j = j$ and $j \geq 1$ (after defining $\bar{\Phi}$) and so Miyahara's results may be derived from ours.

II.3. FLICKER NOISE

We shall now investigate the asymptotic behaviour of the spectral density of the process $\eta_t^*[\phi]$, where η_t^* is the stationary Φ'-valued Ornstein-Uhlenbeck process

$$d\eta_t^* = -L'\eta_t^* + dW_t; \quad \eta_0^* \quad N_{-q}(\bar{m},S).$$

We recall that if X is a real-valued wide-sense stationary process with covariance function $\Gamma(h) = \text{Covar}\,(X_t, X_{t+h})$ then the spectral density ρ of X is simply the Fourier transform of Γ :

$$\rho(\nu) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \Gamma(h)e^{i\nu h}dh.$$

Following J.B. Walsh [29] we shall say that X is a flicker noise iff

$$\lim_{\nu\to\infty} \nu^2 \rho(\nu) = \infty$$

and for $\alpha \in (0,2)$ we shall say that X is an $f^{-\alpha}$-noise, iff for some $c \in (0,\infty)$,

$$\lim_{\nu\to\infty} \nu^{\alpha}\rho(\nu) = c.$$

II.3.1. THEOREM

Suppose that $\lambda_1>0$. Let $\phi \in \Phi$. Let ρ denote the spectral density of $\eta_t^*[\phi]$. Then

$$\lim_{\nu \to \infty} \nu^{\alpha} \rho(\nu) = \begin{cases} 0 & \text{if } \alpha \in (0,2) \\ (2\quad)^{-1/2} Q(\phi,\phi) & \text{if } \alpha = 2. \end{cases}$$

So $\eta_\cdot^*[\phi]$ is neither a flicker nor an $f^{-\alpha}$-noise.

<u>PROOF:</u>

$$\eta_t^*[\phi] = \sum_{j=1}^{\infty} \Big[e^{-\lambda_j t} \eta_o[\phi_j] + m[\phi_j]\lambda_j^{-1}(1-e^{-\lambda_j t})$$

$$+ \int_0^t e^{-\lambda_j (t-s)} dW_s^j[\phi_j] \Big] \langle \phi,\phi_j \rangle_H \quad \text{where}$$

$\eta_o^*[\phi_j] = N(\lambda_j^{-1} m[\phi_j], B(\phi_j,\phi_j))$ and hence

$\mathrm{Covar}(\eta_t^*[\phi], \eta_{t+h}^*[\phi]) =$

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle \phi,\phi_j \rangle_H \langle \phi,\phi_k \rangle_H B(\phi_j,\phi_k) \cdot \begin{cases} e^{-\lambda_j h}; & h \geq 0 \\ e^{-\lambda_k |h|}; & h < 0 \end{cases}$$

The series

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle \phi,\phi_j \rangle_H \langle \phi,\phi_k \rangle_H B(\phi_j,\phi_k)$$

is absolutely convergent for any $\phi \in \Phi$ and therefore

$$\rho(\nu) = \int_{-\infty}^{\infty} (2\pi)^{-1/2} \mathrm{Covar}(\eta_t^*[\phi], \eta_t^*[\phi]) e^{ih\nu}\, dh$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle\phi,\phi_j\rangle_H \langle\phi,\phi_k\rangle_H\, B(\phi_j,\phi_k)\Big(\int_0^{\infty} e^{-\lambda_j h + i\nu h}\, dh +$$

$$\int_{-\infty}^{0} e^{-\lambda_k |h| + i\nu h}\, dh\Big).(2\pi)^{-1/2}$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle\phi,\phi_j\rangle_H \langle\phi,\phi_k\rangle_H\, B(\phi_j,\phi_k)(2\pi)^{-1/2}$$

$$\Big(\frac{-1}{-\lambda_j + i\nu} + \frac{1}{\lambda_k + i\nu}\Big)$$

(19) $$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle\phi,\phi_j\rangle_H \langle\phi,\phi_k\rangle_H\, B(\phi_j,\phi_k)(2\pi)^{-1/2}$$

$$\Big(\frac{\lambda_k - i}{\lambda_k^2 + \nu^2} + \frac{\lambda_j + i}{\lambda_j^2 + \nu^2}\Big).$$

For $\alpha \in (0,2]$,

$$\nu^{\alpha} \left| \frac{\lambda_k - i}{\lambda_k^2 + \nu^2} + \frac{\lambda_j + i}{\lambda_j^2 + \nu^2} \right| = \nu^{\alpha} \left[\frac{(\lambda_j + \lambda_k)^2}{(\lambda_j^2 + \nu^2)(\lambda_k^2 + \nu^2)} \right]^{1/2}.$$

$$\left[\frac{\lambda_j^2 \lambda_k^2 + \nu^4 + 2\lambda_j \lambda_k \nu^2 + (\lambda_j - \lambda_k)^2}{\lambda_j^2 \lambda_k^2 + \nu^4 + (\lambda_j^2 + \lambda_k^2)\nu^2} \right]^{1/2}$$

(20) $$\xrightarrow[\nu \to \infty]{} \begin{cases} 0 \text{ if } \alpha \in (0,2) \\ \lambda_j + \lambda_k \text{ if } \alpha = 2. \end{cases}$$

For $\alpha=2$ we have

$$\sup_{\nu\in\mathbb{R}} \left[\frac{(\lambda_j+\lambda_k)^2}{(\lambda_j^2+\nu^2)(\lambda_k^2+\nu^2)}\right]^{1/2} =$$

$$\lim_{\nu\to\infty} \left[\frac{(\lambda_j+\lambda_k)^2}{(\lambda_j^2+\nu^2)(\lambda_k^2+\nu^2)}\right]^{1/2} = \lambda_j+\lambda_k$$

whereas for $\alpha\in(0,2)$ the supremum is attained for

$$\nu^2 = \frac{(\alpha-1)(\lambda_j^2+\lambda_k^2) + \left[(\alpha-1)^2(\lambda_j^2+\lambda_k^2)^2 + 4(2-\alpha)\alpha\lambda_j^2\lambda_k^2\right]^{1/2}}{2-\alpha}$$

$$=: \gamma_{jk}$$

A short evaluation will show that for all k,j and

$$\frac{\lambda_j^2\lambda_k^2 + \nu^4 + 2\lambda_j\lambda_k\nu^2 + (\lambda - \lambda_k)^2}{\lambda_j^2\lambda_k^2 + \nu^4 + (\lambda_j^2 + \lambda_k^2)\nu^2} \leq$$

$$1 + 2\lambda_1^{-2} =: C$$

Noting that $\gamma_{jk} > 0$ and that $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ we get for $\alpha\in(0,2)$

$$\sup_{\nu\in\mathbb{R}} \left(\frac{(\lambda_j+\lambda_k)^2}{(\lambda_j+\nu^2)(\lambda_k+\nu^2)}\right)^{1/2} =$$

$$\left[(\alpha-1)(\lambda_j^2+\lambda_k^2) + ((\alpha-1)(\lambda_j^2+\lambda_k^2)^2 + 4\alpha(2-\alpha)\lambda_k^2\lambda_j^2)^{1/2}\right]^{\alpha/2}.$$

$$(2-\alpha)^{\alpha/2}\left[\frac{(\lambda_j+\lambda_k)^2}{(\lambda_j^2+\gamma_{jk})(\lambda_k^2+\gamma_{jk})}\right]^{1/2}$$

$$\leq \lambda_1^{-2}(\lambda_j+\lambda_k)\left[\frac{\lambda_j^2+\lambda_k^2+((2\lambda_j^2+2\lambda_k^2)^2)^{1/2}}{2-}\right]^{\alpha/2}$$

$$\leq \lambda_1^{-2}(\lambda_j+\lambda_k)(\lambda_j^2+\lambda_k^2)^{\alpha/2}\left(\frac{3}{2-\alpha}\right)^{\alpha/2}$$

$$\leq \lambda_1^{-2}\left(\frac{3}{2-\alpha}\right)^{\alpha/2}(\lambda_j+\lambda_k)^{\alpha+1}$$

$$\leq \lambda_1^{-2}\left(\frac{3}{2-\alpha}\right)^{\alpha/2}(1+\lambda_j)^{1+\alpha}(1+\lambda_k)^{1+\alpha}$$

and therefore we find

$$\nu^{\alpha}\rho(\nu) \leq$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\langle\phi,\phi_j\rangle_o\ \langle\phi,\phi_k\rangle_o\ B(\phi_j,\phi_k)|\ \nu^{\alpha}\left|\frac{\lambda_k-i}{\lambda_k^2+\nu^2}+\frac{\lambda_j+i}{\lambda_j^2+\nu^2}\right|$$

$$\leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\langle\phi,\phi_j\rangle_o\ \langle\phi,\phi_k\rangle_o\ \frac{Q(\phi_j,\phi_k)}{\lambda_j+\lambda_k}|C\ \cdot$$

$$\begin{cases}\lambda_1^{-2}\left(\frac{3}{2-\alpha}\right)^{\alpha/2}(1+\lambda_j)^{1+\alpha}(1+\lambda_k)^{1+\alpha} & \text{for } \alpha\in(0,2)\\[2ex] \lambda_j+\lambda_k & \text{for } \alpha=2\end{cases}$$

$$\leq \begin{cases} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |<\phi,\phi_j>_o <\phi,\phi_k>_o| \theta_2 (1+\lambda_j)^{r_2} (1+\lambda_k)^{r_2}\ 1/2\lambda_1^{-3} \\ \quad C\left(\frac{3}{2-\alpha}\right)^{\alpha/2} (1+\lambda_j)^{1+\alpha} (1+\lambda_k)^{1+\alpha} \text{ for } \alpha \in (0,2) \\ \\ \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |<\phi,\phi_j>_o <\phi,\phi_k>_o| \theta_2 (1+\lambda_j)^{r_2+1+\alpha} (1+\lambda_k)^{r_2+1+\alpha} \\ \quad C\frac{1}{2\lambda_1} \text{ for } \alpha = 2 \end{cases}$$

$$\leq \begin{cases} C/2\lambda_1^{-3}\theta_2\left(\frac{3}{2-\alpha}\right)^{\alpha/2} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |<\phi,\phi_j>_o| (1+\lambda_j)^{r_1+r_2+1+\alpha} \\ (1+\lambda_j)^{-r_1} |<\phi,\phi_k>_o| (1+\lambda_k)^{r_1+r_2+1+\alpha} (1+\lambda_j)^{-r_1} \quad \text{for} \\ \alpha \in (0,2) \\ \\ \frac{C\theta_2}{2\lambda_1} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |<\phi,\phi_j>| (1+\lambda_j)^{r_1+r_2+\ +1} (1+\lambda_j)^{-r_1} |<\phi,\phi_k>| \\ (1+\lambda_k)^{r_1+r_2+\alpha+1} (1+\lambda_j)^{-r_1} \quad \text{for } \alpha = 2 \end{cases}$$

$$\leq \begin{cases} \frac{C\theta_2}{2\lambda_1^3}\left(\frac{3}{2-\alpha}\right)^{\alpha/2} \theta_1 \|\phi\|^2_{r_1+r_2+1+\alpha} & \text{for } \alpha \in (0,2) \\ \frac{C\theta_2}{2\lambda_1} \theta_1 \|\phi\|_{r_1+r_2+1+\alpha} & \text{for } \alpha = 2. \end{cases}$$

Combining this with (20) the DCT gives

$$\lim_{\nu\to\infty} \nu^{\alpha}\rho(\nu) = \begin{cases} 0 \text{ if } \alpha \in (0,2) \\ b \text{ if } \alpha = 2, \end{cases}$$

where

$$b = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle\phi,\phi_j\rangle \langle\phi,\phi_k\rangle B(\phi_j,\phi_k) (2\pi)^{-1/2} \cdot$$

$$\lim_{\nu\to\infty} \nu^2\left(\frac{\lambda_k - i}{\lambda_k^2+\nu^2} + \frac{\lambda_j+i}{\lambda_j^2+\nu^2}\right)$$

$$= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle\phi,\phi_j\rangle \langle\phi,\phi_k\rangle B(\phi_j,\phi_k)(2\pi)^{-1/2}(\lambda_j+\lambda_k)$$

$$= (2\pi)^{-1/2}Q(\phi,\phi).$$

||

—The assumption that $\lambda_1 > 0$ serves the purpose of assuring that none of the coordinate processes $\eta_t^*[\phi_j]$ is a white noise. The theorem implies that $\rho(\nu)\approx \nu^{-2}$ for large ν. For a one-dimensional Ornstein-Uhlenbeck process the spectral density is proportional to $(\lambda^2 + \nu^2)^{-1}$. In view of this, the conclusion of the theorem is hardly surprising.

Let us look at an example studied by J.B. Walsh [29]:

Take $H = \Phi_0 = L^2([0,b])$ and $L = I - \frac{d^2}{dx^2}$ with Neumann boundary at zero and b. In this case the eigensystem is

$$\phi_j(x) = \begin{cases} b^{-1/2} \text{ for } j = 0 \\ 2^{1/2}b^{-1/2} \cos(\pi j x b^{-1}); \quad j \geq 1 \end{cases} \quad \text{and}$$

$\lambda_j = 1 + \pi^2 j^2 b^{-2}; \quad j = 0,1,\ldots,$

and if, for a given $\sigma^2 \geq 0$, we take

$Q(\phi,\psi) = \sigma^2 \int_0^b \phi(x)\psi(x)dx$ the series

$$\sum_{j=0}^{\infty} \eta_t^*[\phi_j]\phi_j(x)$$

converges for $x \in [0,b]$ (P–a.s.) to a limit $V(t,x)$ satisfying

$$\eta_t^*[\phi] = \int_0^b V(t,x)\phi(x)dx \quad \text{(P–a.s.)}$$

Walsh then showed that for each $x \in [0,b]$, $V(t,x)$ is a flicker noise and that the asymptotic behaviour of its spectral density is that of an $f^{-3/2}$ noise ([29], theorem 8.1.). This result may be obtained from our framework as follows:

when $Q(\phi,\psi) = \sigma^2 \int_0^b \phi(x)\psi(x)dx \quad \sigma^2 \geq 0$ we have

$Q(\phi_j,\phi_k) = \delta_{jk}\sigma^2$, and inserting this in (17) we get:

spectral density of $\eta_t^*[\phi] =$

$$\rho(\nu) = \sum_{j=0}^{\infty} \langle\phi,\phi_j\rangle_o^2 \, (2\pi)^{-1/2} \, \sigma^2 \frac{1}{\lambda_j^2 + \nu^2}.$$

Let $x \in [0,b]$ and let $\phi^{\epsilon}(y)$ be a smooth approximate identity centered at x.

Then $V(t,x) = \lim_{\epsilon \downarrow 0} \eta_t^*[\phi^{\epsilon}]$ and

$$\lim_{\epsilon \downarrow 0} \rho(\nu) = \sum_{j=0}^{\infty} (\phi_j(x))^2 (2\pi)^{-1/2} \sigma^2 \frac{1}{\lambda_j^2 + \nu^2} \tag{21}$$

(note that $\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2 + \nu^2} < \infty$; since $\lambda_j = 1 + \pi^2 j^2 b^{-2}$.

Moreover $|\langle\phi^{\epsilon},\phi_j\rangle_o| \leq 1 \quad \forall \, \epsilon > 0$).

Inserting the expressions for ϕ_j and λ_j in (21) above we get:

spectral density of $V(t,x) = F(\nu) =$

$$\frac{\sigma^2}{b(2\pi)^{1/2}} \left[\frac{1}{1 + \nu^2} + \sum_{j=1}^{\infty} \frac{2\cos^2(k\pi x b^{-1}}{1 + \;\; ^2 j^2 b^{-2} + \nu^2} \right],$$

which (apart from a constant arising from a different normalization of the Fourier-transform) is Walsh's expression. His procedure may now be followed to conclude that $F(\nu) \approx \nu^{-3/2}$ for ν large.

Walsh remarks that the sample paths of V(t,x) for each x are very irregular. Since $\eta_t^*[\phi] = \int_0^b V(t,x)\phi(x)dx$ we would expect the sample paths of $\eta_t^*[\phi]$ to be much smoother than those of $V(t,x)$. The fact that $\rho(\nu)$ is not a flicker noise, whereas $F(\nu)$ is, is therefore intuitively agreeable.

CHAPTER III

LINEAR SDE'S ON A COUNTABLY HILBERT NUCLEAR SPACE: EXISTENCE, UNIQUENESS AND WEAK CONVERGENCE OF SOLUTIONS

We have previously investigated various properties of the solution to a linear Φ'-valued SDE of the form

$$d\eta_t = -L'\eta_t dt + dW_t$$

where $-L$ was the generator of a selfadjoint contraction semigroup $\{T_t : t \geq 0\}$ on a certain Hilbert space H with the property that there exists $r_1 > 0$ such that $(I + L)^{-r_1}$ is Hilbert-Schmidt on H, and where the nuclear space Φ was defined by

$$\Phi = \{\phi \in H : \|(I + L)^q \phi\|_H < \infty \quad \forall\, q \in \mathbb{R}\}.$$

W_t was then a Φ'-valued Wiener Process.

The existence and uniqueness of solutions in the above context (for Wiener and Poisson generated noise) is due to Kallianpur & Wolpert [14]. However, it is also important to be able to solve such linear Φ'-valued SDE's in situations where $\{T_t : t \geq 0\}$ does not have the property that some power of its resolvent is Hilbert-Schmidt and the topology of the nuclear space is not so intimately related to the generator $-L$. Also, it is of interest to be able to solve such equations when the noise is a generel L^2-semimartingale on Φ' (see page 81 for definition).

In section 1 we shall address the question of existence and uniqueness of solutions to SDE's of the form

$$(1) \qquad d\eta_t = A'\eta_t dt + dM_t; \; \eta_o = \eta$$

defined on a generel rigged Hilbert space $\Phi \hookrightarrow H \hookrightarrow \Phi'$ (see Gel'fand & Vilenkin [6] page 106 or Appendix) where $A : \Phi \to \Phi$ is continuous, and A is assumed to coincide on Φ with the generator of a semigroup $\{T_t : t \geq 0\}$ defined on H and mapping Φ into itself. (see AS.1 page 64 for the precise assumptions on A and $\{T_t : t \geq 0\}$), and where M_t is a (weak) Φ'-valued L^2-semimartingale, defined on page 81.

By analogy with the finite dimensional situation we might expect to be able to write the solution as

$$\eta_t = T'_t\eta + \int_0^t T'_{t-s}dM_s$$

which requires a definition and study of Φ'-valued stochastic integrals. Although stochastic calculus has been developed recently by A. S. Ustunel [26], [27], [28] and Korezlioglu & Martias [16] for the dual of a nuclear space, from a user's point of view it is preferable to be able to solve Φ'-valued SDE's without first having to learn stochastic calculus on Φ'. Moreover, since the equation is linear we would suspect it should be solvable without any reference to stochastic calculus. Indeed, by formally applying Itô's lemma to

$\int_0^t T'_{t-s}dM_s$, we get

$$(2) \qquad \eta_t = T'_t\eta + \int_0^t A'T'_{t-s}M_s ds + M_t \quad \text{a.s.}$$

as a candidate for the solution.

In order to show that (2) is indeed the solution to (1) we first show that the stochastic integral equation

$$\xi_t = \eta + \int_0^t A'\xi_s ds + X_t \quad \text{a.s.}$$

has a unique "weakly CADLAG" solution for X in a class of Φ'-valued processes which contains the Φ'-valued L^2-semimartingales and that this solution is given by

(3) $\xi_t = T'_t\eta + \int_0^t A'T'_{t-s}X_s ds + X_t$ a.s.

(this, of course, will include a proof that the right hand side of (3) actually defines a Φ'-valued process).

Once this is established it will follow that (2) is the unique weakly CADLAG solution to (1), and it is then proved that for every T>0 there is $p_T \in \mathbb{N}_o$ such that

$(\eta_t)_{o\leq t\leq T} \in D([0,T],\Phi_{-p_T})$;

the Skorohod space of all Φ_{-p_T}-valued CADLAG mappings on [0,T].

Finally, in section 2 we prove the main result which, loosely speaking, asserts that if the initial condition η and the noise M in (1) converge weakly then so does the solution to (1).

III.1. EXISTENCE AND UNIQUENESS OF SOLUTIONS

Let $\Phi \hookrightarrow H \hookrightarrow \Phi'$ be a real rigged Hilbert space where H is a real separable Hilbert space. Let τ denote the nuclear topology on Φ and let $\{\Phi_r : r \in \mathbb{N}_o\}$ denote the generating sequence of Hilbert spaces for (Φ, τ) and let $\Phi_{-r} := \Phi'_r$

with the strong topology. For $r \in \mathbb{N}_0$, $\|\cdot\|_r$ ($\|\cdot\|_{-r}$) denotes the Hilbert norm on Φ_r (Φ_{-r}). We shall denote by σ the strong topology of Φ' and we recall that (Φ',σ) is the (strict) inductive limit of $(\Phi_{-r}, \|\cdot\|_{-r})_{r\in\mathbb{N}_0}$. $\mathcal{B}(\Phi')$ will denote the σ-field generated by the strongly open sets in Φ'.

To avoid confusion with inner products we shall adopt the notation that for $\eta \in \Phi'$, $\phi \in \Phi$ $\eta[\phi]$ will denote the value of the functional η evaluated at ϕ.

Throughout the rest of this chapter A will denote a τ-continuous linear operator: $\Phi \to \Phi$ satisfying

<u>AS.1.</u> There exists a strongly continuous semigroup $\{T_t : t \geq 0\}$ on H whose generator coincides with A on Φ and such that:

(a) $T_t \Phi \subset \Phi \quad \forall\ t > 0$

(b) $T_t|_{\Phi} : \Phi \to \Phi$ is continuous in (Φ, τ) $\forall\ t > 0$

(c) $t \to T_t\ \phi$ is τ-continuous for every $\phi \in \Phi$.

<u>III.1.1. LEMMA</u>

For any $t \geq s \geq 0$, any $\phi \in \Phi$ and any $F \in \Phi'$ we have

$$(4) \qquad F[T_{t-s}\phi] - F[\phi] = \int_s^t F[T_{u-s}A\phi]du$$

$$(5) \qquad = \int_s^t F[t_{t-u}A\phi]du$$

PROOF:

Let $t \geq s \geq 0$, $\phi \in \Phi$, $F \in \Phi'$. AS.1 (a), (b) and (c) imply that $T_t|_{\Phi}$ is a strongly continuous semigroup on $(\Phi,\mathcal{T})$. Let B denote its generator (wrt. the $\mathcal{T}$-topology) and put

Ψ = Dom(B).

Then Ψ is dense in Φ, and for every $\psi \in \Psi$ we have, since $F \in \Phi'$,

$$\frac{d}{du} F[T_{u-s}\psi] = F[T_{u-s}B\psi]; \quad u > s \text{ and hence}$$

$$F[t_{t-s}\psi] - F[\psi] = \int_s^t F[T_{u-s}B\psi]du \quad \forall\ \psi \in \Psi, \text{ and similarly}$$

$$F[t_{t-s}\psi] - F[\psi] = \int_s^t F[T_{t-u}B\psi]ds \quad \forall\ \psi \in \Psi$$

also, for every $\psi \in \Psi$,

$$\lim_{h\downarrow 0} \frac{(T_h - I)\psi}{h} = B\psi \quad \text{in } (\Phi,\mathcal{T})$$

since $\Phi \hookrightarrow H \hookrightarrow \Phi'$ is a rigged Hilbert space, $\|\cdot\|_H$ is $\mathcal{T}$-continuous and hence

$$\lim_{h\downarrow 0} \left\| \frac{(T_h - I)\psi}{h} - B\psi \right\|_H = 0$$

but, by AS.1 we have for any $\phi \in \Phi$

$$\lim_{h \downarrow 0} \left\| \frac{(T_h - I)\phi}{h} - A\phi \right\|_H = 0$$

and since $\Psi \subset \Phi$ we must have

$$B\psi = A\psi \quad \forall\ \psi \in \Psi, \text{ hence}$$

$$F[T_{t-s}\psi] - F[\psi] = \int_s^t F[T_{t-u}A\psi]du \quad \forall\ \psi \in \Psi$$

$$= \int_s^t F[T_{t-u}A\psi]du \quad \forall\ \psi \in \Psi$$

Now, let $\psi_n \xrightarrow[n\to\infty]{} \phi$ in (Φ,τ), $\psi_n \in \Psi$. Then

$$|F[T_{u-s}A\psi_n] - F[T_{u-s}A\phi]| \xrightarrow[n\to\infty]{} 0$$

for every $u \in [s,t]$, by AS.1. and the fact that $F \in \Phi'$. Further, since $F \in \Phi'$, $F \in \Phi_{-q}$ for some $q \in \mathbb{N}_o$, and since by AS.1. (c) the mapping :

$$[s,t] \ni u \to \|T_{u-s}A\psi_n\|_q$$

is continuous for each $n \in \mathbb{N}$,

$$f(u) := \sup_{n\in\mathbb{N}} \|T_{u-s}A\psi_n\|_q; \ u \in [s,t]$$

defines a lower-semicontinuous function f on [s,t] (note that the above supremum is finite for each $u \in [s,t]$, since $\phi_n \to \phi$ in (Φ, τ)). In particular, f is bounded on $[s,t]$, and

$$|F[T_{u-s}A\phi_n]| \leq \|F\|_{-q}\|T_{u-s}A\phi_n\|_q$$

$$\leq \|F\|_{-q} f(u) \quad \forall\, n \in \mathbb{N}$$

and hence the DCT gives

$$\int_s^t F[T_{u-s}A\phi_n]du \xrightarrow[n\to\infty]{} \int_s^t F[T_{u-s}A\phi]ds, \text{ but}$$

$$\int_s^t F[T_{u-s}A\phi_n]du = F[T_{u-s}\phi_n] - F[\phi_n]$$

$$\xrightarrow[n\to\infty]{} F[T_{t-s}\phi] - F[\phi], \text{ by AS.1. (b)}$$

and hence

$$F[T_{t-s}\phi] - F[\phi] = \int_s^t F[T_{u-s}A\phi]du$$

In a similar way we obtain

$$F[T_{t-s}\phi] - F[\phi] = \int_s^t F[T_{t-u}A\phi]du$$

Since $\phi \in \Phi$ was arbitrary, the proof is complete.

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III.1.2. THEOREM

For any $\eta_o \in \Phi'$ there is a unique Φ'-valued weakly differentiable function $\eta : [0,\infty] \to \Phi'$ satisfying

$$\left.\begin{aligned} \frac{d}{dt}\eta(t)[\phi] &= \eta(t)[A\phi] \\ \eta(0)[\phi] &= \eta_o[\phi] \end{aligned}\right\} \quad \forall\ \phi \in \Phi$$

PROOF:

EXISTENCE: Let T_t' denote the adjoint of T_t regarded as a bounded linear operator: $\Phi \to \Phi$. We claim that

$\eta(t) := T_t'\eta_o$ is a solution:

In view of AS.1. (a), (b) and (c), let B denote the generator of $\{T_t : t > 0\}$ wrt. the τ-topology and let Ψ = Domain(B). As we have seen previously, $B\psi = A\psi\ \forall\ \psi \in \Psi$. Now, for $\psi \in \Psi$, $t \to T_t'\eta_o[\psi]$ is differentiable with

$$(*) \qquad \frac{d}{dt}\eta_o[T_t\psi] = \eta_o[T_tB\psi] = \eta_o[T_tA\psi].$$

As a consequence of AS.1. (c) $t \to T_t'\eta_o$ is weakly continuous, so for any T>0,

$$\sup_{0\leq t\leq T} |T_t'\eta_o[\phi]| < \infty \quad \forall\ \phi \in \Phi.$$

Hence the Banach-Steinhaus theorem yields the existence of $p_T \in \mathbb{N}_o$ and a constant $C_T>0$ such that

$$\sup_{0\leq t\leq T} |T_t'\eta_o[\phi]| \leq C_T\|\phi\|_{p_T} \quad \forall\ \phi \in \Phi$$

But then $T_t'\eta_o \in \Phi_{-p_T} \quad \forall\ t \in [0,T]$ and

$$\sup_{0\leq t\leq T} \|T_t'\eta_o\|_{-p_T} \leq C_T < \infty$$

Now, fix $\phi \in \Phi$, and let $\psi_n \in \Phi$; $\psi_n \to \phi$ in (Φ,τ) as $n \to \infty$. Then, for any $T>0$,

$$\sup_{0\leq t\leq T} |\eta_o[T_t\psi_n] - \eta_o[T_t\phi]| \leq C_T\|\psi_n-\phi\|_{p_T} \to 0 \text{ as } n \to \infty$$

Hence $t \to \eta_o[T_t\phi] = T_t'\eta_o[\phi]$ is differentiable and (*) and AS.1. (b) now give

$$\frac{d}{dt}\eta_o[T_t\phi] = \lim_{\psi_n\to\phi} \eta_o[T_tA\psi_n]$$

$$= \eta_o[T_tA\phi],$$

concluding the proof of existence.

<u>UNIQUENESS:</u> Suppose that $\xi(t)$ is another Φ'-valued weakly differentiable solution. Let $y(t) := \eta(t) - \xi(t)$. Then $y(t)$ is weakly differentiable and satisfies

$$\frac{d}{dt}y(t)[\phi] = y(t)[A\phi] \qquad \forall\ \phi \in \Phi$$

$$y(0)[\phi] = 0$$

For each $t>0$ let $z(s) := T'_{t-s}y(s)$; $s \in [0,t]$. Then there is a dense set $\tilde{\Phi}$ Φ such that $[0,t)$ $s \to z(s)[\psi]$ is differentiable for each $\psi \in \tilde{\Phi}$, and

$$\frac{d}{ds}z(s)[\psi] = 0 \quad \forall\ \psi \in \tilde{\Phi} \quad \forall\ s \in (0,t)$$

[Proof: Combining AS.1. (a), (b) and (c) we see that $T_t|_{\Phi}$ is a strongly (i.e. $\tau-$) continuous semigroup of linear operators on Φ. Let B denote its generator (wrt. the τ-topology) and take $\tilde{\Phi}$ = Dom(B). Since $\|.\|_H$ is τ-continuous it follows that

$A\psi = B\psi \quad \forall\ \psi \in \tilde{\Phi}$. Fix $s \in (0,t)$. Then for any $\psi \in \tilde{\Phi}$ we have

$$\left|\frac{z(s+h)[\psi]-z(s)[\psi]}{h}\right| =$$

$$\left|\frac{y(s+h)[T_{t-s-h}\psi]-y(s)[T_{t-s}\psi]}{h}\right| =$$

$$\left|y(s+h)\left[\frac{T_{t-s-h}\psi-T_{t-s}\psi}{h}\right] + \frac{y(s+h)[T_{t-s}\psi]-y(s)[T_{t-s}\psi]}{h}\right|$$

But $y(.)[\phi]$ is differentiable for all $\phi \in \Phi$, so

$$\lim_{h\to 0} \frac{y(s+h)[T_{t-s}\psi] - y(s)[T_{t-s}\psi]}{h} =$$

$$\frac{d}{du}y(u)[T_{t-s}\phi]\Big|_{u=s} = y(s)[AT_{t-s}\phi]$$

$$= y(s)[T_{t-s}A\phi]$$

$$(7) \qquad = y(s)[T_{t-s}B\phi]$$

Further, since $u \to y(u)$ is weakly continuous, we have for any compact set K with $s \in \text{interior}(K)$

$$\sup_{h\in K} |y(s+h)[\phi]| < \infty \quad \forall\, \phi \in \Phi,$$

and therefore the Banach-Steinhaus theorem yields the existence of a constant C_K and $r_K \in \mathbb{N}_o$ such that

$$\sup_{h\in K} |y(s+h)[\varphi]| \leq C_K \|\phi\|_{r_K} \quad \forall\, \phi \in \Phi$$

But then for $s+h \in K$:

$$\left|y(s+h)\left[\frac{T_{t-s-h}\phi - T_{t-s}\phi}{h} + T_{t-s}B\phi\right]\right| \leq$$

$$C_K \left\| \frac{T_{t-s-h}\phi - T_{t-s}\phi}{h} + T_{t-s}B\phi \right\|_{r_K}$$

$\to 0$ as $h \to 0$ since $\phi \in \Phi$ and $\|.\|_r$ is τ-continuous $\forall\, r \in \mathbb{N}_o$. Thus

$$\lim_{h\to 0} y(s+h)\left[\frac{T_{t-s-h}\phi - T_{t-s}\phi}{h}\right] = -y(s)[T_{t-s}B\phi],$$

and combining this with (7) we get

$\lim_{h\to 0} \left|\frac{z(s+h)\phi - z(s)\phi}{h}\right| = 0$ as desired.]

Therefore for any $\Delta \in (0,t)$ we have for every $\phi \in \Phi$

$$z(t)[\phi] - z(\Delta)[\phi] = \int_\Delta^t \frac{d}{ds} z(s)[\phi]ds$$

$$= 0, \text{ and hence}$$

$$y(t)[\phi] = z(t)[\phi] = z(\Delta)[\phi] \quad \forall\ \phi \in \Phi,\ \forall\ \Delta \in (0,t) \text{ so}$$

(8) $$y(t)[\phi] = y(\Delta)[T_{t-\Delta}\phi] \quad \forall\ \phi \in \Phi,\ \forall \Delta \in (0,t)$$

But $\Delta \to y(\Delta)$ is weakly continuous, so again the Banach–Steinhaus theorem yields the existence of a constant C_t and $r_t \in \mathbb{N}_0$ such that

$$\sup_{0\leq\Delta\leq t} |y(\Delta)[\phi]| \leq C_t \|\phi\|_{r_t} \quad \forall\ \phi \in \Phi$$

Hence $y(\Delta) \in \Phi_{-r_t} \quad \forall \Delta \in [0,t]$

Now let $y_n := y(\frac{t}{n})$; $n\geq 2$. Then, since $y(0) = 0$, y_n is weakly convergent to zero in Φ' and hence strongly convergent to zero in Φ' (see e.g. Gel'fand & Vilenkin [6] page 73), and $y_n \in \Phi_{-r_t}\ \forall\ n\geq 2$. But the strong topology of Φ' induces the $\|\cdot\|_{-r_t}$ – topology on Φ_{-r_t}. Hence $y_n \xrightarrow[n\to\infty]{} 0$ in Φ_{-r_t}. By (8) we have

$$y(t)[\phi] = y_n[T_{t-\frac{t}{n}}\phi] \quad \forall\ \phi \in \Phi, \quad \forall\ n>2 \text{ so}$$

$$|y(t)[\phi]| \leq \|y_n\|_{-r_t} \|T_{t-\frac{t}{n}}\, \phi\|_{r_t} \quad \forall\ \phi \in \boldsymbol{\Phi},\ \forall\ n \geq 2$$

and letting $n \to \infty$ we get (since
$\|T_{t-\frac{t}{n}}\, \phi\|_{r_t} \to \|T_t \phi\|_{r_t}$ as $n \to \infty$)

$$y(t)[\phi] = 0 \quad \forall\ \phi \in \boldsymbol{\Phi}$$

Since $\boldsymbol{\Phi}$ is dense in Φ and $y(t) \in \Phi'$ it follows that $y(t) = 0$. But $t>0$ was arbitrary. Hence $y(t) = 0\ \forall\ t>0$, concluding the proof of uniqueness.

Let (Ω, F, P) be a complete probability space. In the sequel all stochastic processes and random variables will be defined on (Ω, F, P).

A mapping $Y : \Omega \to \Phi'$ will be called a Φ'-valued random variable iff Y is $\mathcal{B}(\Phi')/F$ measurable.

A mapping $Y_p : \Omega \to \Phi_{-p}$ will be called a Φ_{-p}-valued random variable iff Y_p is $\mathcal{B}(\Phi_{-p})/F$ measurable; where $\mathcal{B}(\Phi_{-p})$ denotes the Borel σ-field on Φ_{-p}, $p \in \mathbb{N}_0$.

Let $I \subset [0,\infty)$. A mapping $X : I \times \Omega \to \Phi'$ (respectively $I \times \Omega \to \Phi_{-p}$) will be called a Φ'-valued (respectively Φ_{-p}-valued) (stochastic) process iff $\forall\ t \in I$ $X_t(.)$ is a Φ'-valued (respectively Φ_{-p}-valued) random variable.

A Φ'—valued process $X = (X_t)_{t \in I}$ will be called measurable iff $(t,\omega) \to X_t(\omega)$ is $\mathcal{B}(\Phi')/\mathcal{B}(I) \times \mathcal{F}$ measurable; where $\mathcal{B}(I)$ is the Borel σ—field on I.

Similarly, a Φ_{-p}—valued process $X = (X_t)_{t \in I}$ will be called measurable iff $(t,\omega) \to X_t(\omega)$ is $\mathcal{B}(\Phi_{-p})/\mathcal{B}(I) \times \mathcal{F}$ measurable.

Let $\{X_t : t \geq 0\}$ be a Φ'—valued process satisfying

<u>AS.2.:</u> $\forall\ t>0\ \exists\ \Omega_t \in \mathcal{F}$ with $P(\Omega_t) = 1$, $\exists q(t) \in \mathbb{N}_0$ such that

$$X_s(\omega) \in \Phi_{-q(t)} \quad \forall\ s \in [0,t] \quad \forall\ \omega \in \Omega_t \text{ and}$$

$\forall\ \omega \in \Omega_t$: the mapping $[0,t] \ni s \to X_s(\omega)$ is CADLAG wrt. $\|\cdot\|_{-q(t)}$

Then, for every $t>0$ $X^t := (X_s)_{s \in [0,t]}$ is a $\Phi_{-q(t)}$—valued $\|\cdot\|_{-q(t)}$—CADLAG process (P.a.s.) and therefore (since $(\Phi_{-q(t)}, \|\cdot\|_{-q(t)})$ is a complete metric space) X^t is a $\Phi_{-q(t)}$—valued <u>measurable</u> process. Since (Φ',σ) is the (strict) inductive limit of $\{(\Phi_{-q}, \|\cdot\|_{-q})\}_{q \in \mathbb{N}_0}$ it follows that X is a Φ'—valued <u>measurable</u> process.

We can then show:

III.1.3. THEOREM

Let $\{X_t : t \geq 0\}$ be a Φ'-valued process satisfying AS.2..
Let η be a Φ'-valued random variable. Then

(a) There exists a Φ'-valued process ξ_t satisfying

(9) $P\{\xi_t[\phi] = \eta[\phi] + \int_0^t \xi_s[A\phi]ds + X_t[\phi] \ \forall \ \phi \in \Phi\} = 1$
$\forall \ t \geq 0$ and

(10) $\exists \ G \in \mathcal{F}$ with $P(G) = 1$ such that $t \to \xi_t(\omega)$ is CADLAG wrt. the weak topology of Φ' for every $\omega \in G$.

(b) If ξ_t and η_t are two Φ'-valued processes satisfying (9) and (10) then

(11) $P\{\xi_t = \eta_t \quad \forall \ t \geq 0\} = 1.$

PROOF:

(a) EXISTENCE: Fix $t>0$. By AS.2., there exists $\Omega_t \in \mathcal{F}$ with $P(\Omega_t = 1)$ and $q(t) \in \mathbb{N}_0$ such that

$X_s(\omega) \in \Phi_{-q(t)} \quad \forall \ s \in [0,t] \ \forall \ \omega \in \Omega_t$ and

$s \to X_s(\omega)$ is CADLAG wrt. $\|\cdot\|_{-q(t)} \quad \forall \ \omega \in \Omega_t$

But then for $\phi \in \Phi$: $|X_s(\omega)[T_{t-s}A\phi]| \leq$

$$\|X_s(\omega)\|_{-q(t)}\|T_{t-s}A\phi\|_{q(t)} \quad \forall\, s \in [0,t]\ \forall\, \omega \in \Omega_t.$$

AS.1. (c) implies that $s \to \|T_{t-s}A\phi\|_{q(t)}$ is continuous on $[0,t]$ and therefore the integral

$\int_0^t X_s(\omega)[T_{t-s}A\phi]ds$ is finite for all $\omega \in \Omega_t$ and all $\phi \in \Phi$.

We claim that for every $\omega \in \Omega_t$ the map

$\phi \to \int_0^t X_s(\omega)[T_{t-s}A\phi]ds$ is continuous on (Φ,τ):

Let $\omega \in \Omega_t$. Let $\phi_n \to 0$ in (Φ,τ). Then

$$\sup_n \|T_{t-s}A\phi_n\|_{q(t)}<\infty \quad \forall\, s \in [0,t]$$

(since AS.1. implies that $\phi \to \|T_{t-s}A\phi\|_{q(t)}$ is continuous wrt. τ). Define

$$f(s) = \sup_n \|T_{t-s}A\phi_n\|_{q(t)}$$

Since $s \to \|T_{t-s}A\phi\|_{q(t)}$ is continuous on $[0,t]$ by AS.1. (c), f is a lower semicontinuous function on $[0,t]$ and therefore bounded on $[0,t]$. Hence $f \in L^1([0,t])$. Now,

$$\|T_{t-s}A\phi_n\|_{q(t)} \leq f(s) \quad \forall\, n \in \mathbb{N} \text{ and}$$

$\|T_{t-s}A\phi_n\|_{q(t)} \to 0$ as $n\to\infty \quad \forall\, s \in [0,t]$

(recall that A is continuous on Φ and so is T_{t-s} by AS.1.(b)). Since

$$|X_s(\omega)[T_{t-s}A\phi_n]| \le \|X_s(\omega)\|_{-q(t)}\|T_{t-s}A\phi_n\|_{q(t)}$$

$$< \|X_s(\omega)\|_{-q(t)}\, f(s) \quad \forall\, n \in \mathbb{N}$$

and since $s \to \|X_s(\omega)\|_{-q(t)}$ is CADLAG (note that

$$|\,\|X_s(\omega)\|_{-q(t)} - \|X_u(\omega)\|_{-q(t)}| \le \|X_s(\omega)-X_u(\omega)\|_{-q(t)})$$

and therefore in $L^1([0,t])$, the DCT gives

$$\int_0^t X_s(\omega)[T_{t-s}A\phi_n]ds \to 0 \text{ as } n\to\infty.$$

Thus, $\phi \to \int_0^t X_s(\omega)[T_{t-s}A\phi]$ is continuous on Φ for each $\omega \in \Omega_t$. Also, for each $\phi \in \Phi$, the mapping $\Omega_t \ni \omega \to \int_0^t X_s(\omega)[T_{t-s}A\phi]ds$ is measurable since $(X_t)_{t\geq 0}$ is a measurable process.

Now, a Φ'-valued map Y on $(\Omega,\mathcal{F})$ is a Φ'-valued random variable iff $Y[\phi]$ is a real random variable for every $\phi \in \Phi$ (recall that for a countably Hilbert nuclear space Φ the σ-field generated by the strongly open sets in Φ' is the same as the σ-field generated by the weakly open sets in Φ' which in turn is equal to the smallest σ-field in

Φ' with respect to which all the evaluation maps $\phi : \eta \to \eta[\phi]$; $\eta \in \Phi'$, $\phi \in \Phi$, are measurable). Therefore the Φ'-valued map $\xi_t^1 : \Omega \to \Phi'$ given by

$$\xi_t^1(\omega)[\phi] = \begin{cases} \int_0^t X_s(\omega)[T_{t-s}A\phi]ds \text{ for } \omega \in \Omega_t \\ \\ 0 \text{ for } \omega \notin \Omega_t \end{cases}$$

is a Φ'-valued random variable. Now, define a Φ'-valued map $\xi_t : \Omega \to \Phi'$ by

$$\xi_t(\omega) = \begin{cases} T'_t\eta(\omega) + \xi_t^1(\omega) + M_t(\omega); \quad \in \Omega_t \\ \\ 0 \text{ for } \omega \notin \Omega_t \end{cases}$$

(where $T'_t : \Phi' \to \Phi'$ is the adjoint of T_t considered as a continuous linear operator on Φ).
Since X_t and η are Φ'-valued random variables and since T_t satisfies AS.1. (b), ξ_t is a Φ'-valued random variable. Hence $(\xi_t)_{t\geq 0}$ is a Φ'-valued process.

Next, we claim that $\{\xi_t : t\geq 0\}$ satisfies (9): Fix $t>0$ and let $\phi \in \Phi$. Recall (1) of Lemma III.1.1, i.e. for any $F \in \Phi'$ and $0\leq s\leq t$ we have

$$(12) \qquad F[T_{t-s}\psi] - F[\psi] = \int_s^t F[T_{u-s}A\phi]du \quad \forall \psi \in \Phi$$

so, in particular, letting $F = X_s$ and $\psi = A\phi$, we have for $\omega \in \Omega_t$:

$$(13) \qquad X_s(\omega)[T_{t-s}A\phi] = X_s(\omega)[A\phi] + \int_s^t X_s(\omega)[T_{u-s}A^2\phi]du$$

$\forall\ s \in [0,t]$

Each of the following statements holds for every $\omega \in \Omega_t$:

$$\xi_t(\omega)[\phi] = \eta(\omega)[T_t\phi] + X_t(\omega)[\phi] + \int_0^t X_s(\omega)[T_{t-s}A\phi]ds$$

(by (12) applied to $\eta(\omega)$ with $s = 0$)
$\downarrow$

$$= \eta(\omega)[\phi] + \int_0^t \eta(\omega)[T_uA\phi]du + X_t(\omega)[\phi] + \int_0^t X_s(\omega)[T_{t-s}A\phi]ds$$

(by (13))
$\downarrow$

$$= \eta(\omega)[\phi] + \int_0^t \eta(\omega)[T_uA\phi]du + X_t(\omega)[\phi] + \int_0^t \left[X_s(\omega)[A\phi] + \int_s^t X_s(\omega)[T_{u-s}A^2\phi]du\right]ds$$

$$= \eta(\omega)[\phi] + \int_0^t \eta(\omega)[T_uA\phi]du + X_t(\omega)[\phi] + \int_0^t X_u(\omega)[A\phi]du + \int_0^t \int_0^u X_s(\omega)[T_{u-s}AA\phi]dsdu$$

$$= \eta(\omega)[\phi] + \int_0^t \left[\eta(\omega)[T_uA\phi] + X_u(\omega)[A\phi] + \int_0^u X_s(\omega)[T_{u-s}AA\phi]ds\right]du + X_t(\omega)[\phi]$$

$$= \eta(\omega)[\phi] + \int_0^t \xi_u(\omega)[A\phi]du + X_t(\omega)[\phi].$$

Since $\phi \in \Phi$ was arbitrary, (9) is proved. To prove (10) note that AS.1. (b) and the fact that $\eta \in \Phi'$ imply that $t \to \eta(\omega)[T_t\phi]$ is continuous for every $\omega \in \Omega$ and every $\phi \in \Phi$. We shall conclude the proof of (10) by showing that for P.a.s. $\omega \in \Omega$ the mappings $t \to X_t(\omega)[\phi]$ and $t \to \int_0^t X_s(\omega)[T_{t-s}A\phi]$ are resp. CADLAG and continuous for every $\phi \in \Phi$:

Let $T_n \uparrow \infty$. By AS.2. for each $n \in \mathbb{N}$ there is $\Omega_n \in \mathcal{F}$ with $P(\Omega_n) = 1$ and $q_n \in \mathbb{N}$ such that

$$X_t(\omega) \in \Phi_{-q_n} \quad \forall\, t \in [0,T_n] \quad \forall\, t \in \Omega_n$$

and the mapping $t \to X_t(\omega)$ is $\|\cdot\|_{-q_n}$-CADLAG on $[0,T_n]$ for every $\omega \in \Omega_n$.

Let $G_n = \bigcap_{n\geq 1} \Omega_n$. Then $P(G) = 1$ and for each $n \in \mathbb{N}$

$$t \to X_t(\omega) \text{ is } \|\cdot\|_{-q_n}\text{-CADLAG on } [0,T_n].$$

Fix $u\geq 0$. Then $u \in [0,T_{n_o})$ for some $n_o \in \mathbb{N}$ and hence we have

$$\forall\, \omega \in G : |X_t(\omega)[\phi] - X_u(\omega)[\phi]| \leq$$

$$\|X_t(\omega) - X_u(\omega)\|_{-q_{n_o}} \|\phi\|_{q_{n_o}} \quad \forall\, \phi \in \Phi,\ \forall\, t \in [0,T_{n_o}]$$

So, for each $\omega \in G$, $t \to X_t(\omega)[\phi]$ is right continuous at u for every $\phi \in \Phi$. If $u>0$, take $t,s<u$, and we have for each

$\omega \in G$:

$$|X_t(\omega)[\phi] - X_s(\omega)[\phi]| \leq$$

$$\|X_t(\omega) - X_s(\omega)\|_{-q_{n_o}} \|\phi\|_{q_{n_o}} \quad \forall \phi \in \Phi.$$

But the limit as $t,s\uparrow u$ exists and is equal to zero on the right hand side by choice of G. Therefore, for each $\omega \in G$ $t \to X_t(\omega)[\phi]$ is CADLAG at u for every $\phi \in \Phi$. Since $u\geq 0$ was arbitrary, $t \to X_t(\omega)$ is CADLAG wrt. the weak topology of Φ' for every $\omega \in G$. Next, we will show that

$$\forall \ \omega \in G : t \to \int_0^t X_s(\omega)[T_{t-s}A.]ds \in \Phi'$$

is continuous wrt. the weak topology of Φ': Fix $\omega \in G$ and let $u\geq 0$. Then $u \in [0,T_{n_o}]$ for some $n_o \in \mathbb{N}$ and $t \to X_t(\omega)$ is $\|.\|_{-q_{n_o}}$ –CADLAG on $[0,T_{n_o}]$.

Therefore, there exists a constant $L = L(\omega)$ such that

$$\sup_{t\in[0,T_{n_o}]} \|X_s(\omega)\|_{-q_{n_o}} \leq L(\omega).$$

But then, for any $\phi \in \Phi$ and $t \in [0,T_{n_o}]$ we have

$$\left|\int_0^t X_s(\omega)[T_{t-s}A\phi]ds - \int_0^u X_s(\omega)[T_{u-s}A\phi]ds\right| =$$

$$\left|\int_0^{t\wedge u} X_s(\ )[T_{t-s}A\phi - T_{u-s}A\phi]ds + \right.$$

$$\operatorname{sgn}(t-u) \int_{u\wedge t}^{u\vee t} X_s(\omega)[T_{t\vee u-s}A\phi]ds\Big| \leq$$

$$\int_0^{t\wedge u} L(\omega)\,\|T_{t-s}A\phi - T_{u-s}A\phi\|_{q_{n_o}} ds +$$

$$\Big|\int_{t\wedge u}^{t\vee u} X_s(\omega)[T_{t\vee u-s}A\phi]ds\Big| \leq$$

$$\int_0^{T_{n_o}} L(\omega)\,\|T_{t-s}A\phi - T_{u-s}A\phi\|_{q_{n_o}} 1_{[0,t\ u]}(s)ds +$$

$$\int_{t\wedge u}^{t\vee u} L(\omega)\,\|T_{t\vee u-s}A\phi\|_{q_{n_o}} ds.$$

The first term tends to zero as $t \to u$ by the DCT since

$$\|T_{t-s}a\phi - T_{u-s}A\phi\|_{q_{n_o}} \leq 2 \sup_{0\leq s\leq t\leq T_{n_o}} \|T_{t-s}A\phi\|_{q_{n_o}} < \infty$$

(by continuity, since AS.1. (c) implies that

$(s,t) \to \|T_{t-s}A\phi\|_{q_{n_o}}$ is continuous on

$\{(s,t) \in [0,T_{n_o}]^2 : 0\leq s\leq t\}$).

The second term tends to zero as $t \to u$ since

$$\|T_{t\ u-s}A\phi\|_{q_{n_o}} \cdot 1_{[t\ u,t\ u]}(s) \leq$$

$$\sup_{0\leq s\leq t\leq T_{n_o}} \|T_{t-s}A\phi\|_{q_{n_o}} < \infty,$$

and since $t\vee u - t\wedge u = |t-u|$.

As $u \geq 0$ was arbitrary and $\phi \in \Phi$ was arbitrary we see that $t \to \int_0^t X_s(\omega)[T_{t-s}A.]ds$ is continuous wrt. the weak topology on Φ' for every $\omega \in G$.

Thus $t \to \xi_t(\omega)$ is CADLAG wrt. the weak topology of Φ' for every $\omega \in G$. This concludes the proof of (10).

<u>(b)</u> <u>UNIQUENESS:</u> Suppose that ζ_t is another Φ'-valued process satisfying (9) and (10). Let $y(t) = \zeta_t - \xi_t$. Then, for each $t \geq 0$ there is $\Omega_t \in \mathcal{F}$ such that $P(\Omega_t) = 1$ and

$$\forall \omega \in \Omega_t : y(t)(\omega)[\phi] = \int_0^t y(s)(\omega)[A\phi]ds \quad \forall \phi \in \Phi.$$

Also, there is $G \in \mathcal{F}$ with $P(G) = 1$ such that $t \to y(t)(\omega)$ is CADLAG wrt. the weak topology of Φ'. Hence, there is $G_1 \in \mathcal{F}$ with $P'G_1) = 1$ such that

$$y(t)(\omega)[\phi] = \int_0^t y(s)(\omega)[A\phi]ds \quad \forall \phi \in \Phi, \forall t \geq 0$$

for each $\omega \in G_1$. Hence

$$y(t)(\omega)[\phi] = 0 \quad \forall \phi \in \Phi, \forall t \geq 0$$

for each $\omega \in G_1$, by theorem III.1.2. (take $\eta_0 = 0$ in theorem III.1.2. and use the uniqueness part). Thus

$P\{y(t) = 0 \quad \forall t \geq 0\} = 1$, as claimed.

$/\!/$

REMARK 1.

Note that we actually showed that the Φ'-valued process given by

$$\xi_t = T'_t\eta + X_t + \int_0^t A'T'_{t-s}X_s ds$$

(the integral being in the weak sense) is the unique (in the sense of (11)) Φ'-valued stochastic process satisfying (9) and (10).

Note also that we showed that for every $\omega \in G$ the mapping

$$t \to \eta(\omega)[T_t\phi] + \int_0^t X_s(\omega)[T_{t-s}A\phi]ds$$

is continuous for every $\phi \in \Phi$ when $X = (X_s)_{s\geq 0}$ satisfies AS.2..

Let $\{\mathcal{F}_t : t\geq 0\}$ be a right continuous (i.e. $\bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t \ \forall\ t\geq 0$) filtration on $(\Omega,\mathcal{F})$ such that $\mathcal{F}_0$ contains all P-null sets.

Recall that a real-valued $\mathcal{F}_t$-adapted process $M = (M_t)_{t\geq 0}$ is called an L^2-semimartingale wrt. $(\mathcal{F}_t)_{t\geq 0}$ iff M admits a decomposition $M = B + M^1$, where $M^1 = (M^1_t)_{t\geq 0}$ is a CADLAG martingale wrt. $(\mathcal{F}_t)_{t\geq 0}$ satisfying $E(M^1_t)^2 < \infty\ \forall\ t\geq 0$ and $B = (B_t)_{t\geq 0}$ is a CADLAG $\mathcal{F}_t$-adapted process of bounded variation on compact sets satisfying $EB^2_t < \infty\ \forall\ t\geq 0$.

DEFINITION

A Φ'-valued process $M = (M_t)_{t\geq 0}$ is called a (weak) Φ'-valued L^2-semimartingale wrt. $(\mathbb{F}_t)_{t\geq 0}$ iff

$$\forall\, \phi \in \Phi : (M_t[\phi])_{t\geq 0} \text{ is a real-valued } L^2\text{-semimartingale wrt. } (\mathbb{F}_t)_{t\geq 0}.$$

REMARK 2

A. S. Ustunel [26] has defined the notion of a (strong) Φ'-valued semimartingale. A (weak) semimartingale in the above sense gives rise to a strong Φ'-valued semimartingal (see [26], theorem III.1.), whereas if $X = (X_t)_{t\geq 0}$ is a strong Φ'-valued semimartingale then $(X_t[\phi])_{t\geq 0}$ is a real-valued local semimartingale which is not necessarily in $L^2(\Omega,\mathbb{F},P)$, $(\phi \in \Phi)$.

The L^2 property is, however, crucial to our argument.

III.1.4. THEOREM

Let $(X_t : t\geq 0)$ be a Φ'-valued semimartingale (wrt. $(\mathbb{F}_t)_{t\geq 0}$). Then $(X_t : t\geq 0)$ satisfies assumption AS.2.

PROOF:

(Adapted from a proof of I. Mitoma concerning Gaussian processes ([21], theorem 1, proof page 211/212)).

Fix $t>0$. Since $(X_s[\phi] : s \in [0,t])$ is a real-valued L^2-semimartingale for each $\phi \in \Phi$,

$$E(\sup_{0\leq s\leq t} |X_s[\phi]|)^2 < \infty \quad \forall \phi \in \Phi.$$

Since $X_s(\omega)$, $s \in [0,t]$ $\omega \in \Omega$ is a continuous functional on Φ, the mapping $X^t(\omega)$ defined on Φ by

$$X^t(\omega)(\phi) := \sup_{s\in[0,t]} |X_s(\omega)[\phi]|$$

is a lower semicontinuous function of $\phi \in \Phi$ for P-a.s., $\omega \in \Omega$ (note that the above supremum is finite for all $\phi \in \Phi$ and P.a.s. $\omega \in \Omega$, since $s \to X_s(\omega)[\phi]$ is CADLAG for every $\phi \in \Phi$ and P-a.s. $\omega \in \Omega$).

But then $V_t(\phi) := E(X^t(\phi))^2$ is also a lower semicontinuous function of $\phi \in \Phi$, because if $\phi_n \to \phi$ in Φ, then Fatou's lemma gives

$$\begin{aligned}\liminf_{n\to\infty} V_t(\phi_n) &\geq E(\liminf_{n\to\infty}(X^t(\phi_n))^2) \\ &\geq E(X^t(\phi))^2 \\ &= V_t(\phi).\end{aligned}$$

Hence, for any $n \in \mathbb{N}$ the set $\{\phi : V_t(\phi) \leq n\}$ is closed

$(V_t(\phi) \geq 0 \ \forall \ \phi)$. Now,

$\Phi = \bigcup_{n\geq 1} \{\phi : V_t(\phi) \leq n\}$ and since (Φ,τ) is a complete metric space, Baire's theorem implies the existence of $n_o \in \mathbb{N}$ such that interior$(\{\phi : V_t(\phi) \leq n_o\}) \neq \emptyset$, i.e. $\{\phi : V_t(\phi) \leq n_o\}$ contains a τ–neighbourhood of zero in Φ.

But V_t is a convex function of ϕ satisfying $V_t(a\phi) = |a|^2 V_t(\phi) \ \forall \ a \in \mathbb{R}$ and hence

$E_t := \{\phi \in \Phi : V_t(\phi) \leq n_o\}$ is convex and balanced.

Now E_t contains a τ–neighbourhood of zero in Φ (and hence E_t is a'so absorbing), i.e. there is a set D_t of the form

$$D_t = \{\phi \in \Phi : \|\phi\|_{p_t} < \epsilon_t\}; \ \epsilon_t > 0 \quad \text{such that } D_t \subset E_t.$$

But then there is a constant K_1 such that $p_{E_t}(\phi) \leq K_1 p_{D_t}(\phi) \ \forall \ \phi \in \Phi$ (where $p_B(.)$ denotes the Minkowski–functional for the convex, balanced and absorbing set B). Now,

$$p_{E_t}(\phi) = \left(\frac{V_t(\phi)}{n_o}\right)^{\frac{1}{2}} \quad \text{and} \quad p_{D_t}(\phi) = \frac{\|\phi\|_{p_t}}{\epsilon_t} \quad \text{hence}$$

$$V_t(\phi) \leq K_1^2 n_o \epsilon_t^{-2} \|\phi\|_{p_t}^2 \quad \forall \ \phi \in \Phi.$$

Since Φ is countably Hilbert nuclear there is $r_t \geq p_t$ such that the canonical injection $i_{r_t}^{p_t} : \Phi_{r_t} \rightarrow \Phi_{p_t}$ is Hilbert–Schmidt. Let $\{\phi_k : k \in \mathbb{N}\}$ be a CONS in Φ_{r_t} consisting of elements of Φ. Then

$$\sum_{k=1}^{\infty} \|\phi_k\|_{p_t}^2 < \infty, \text{ so}$$

$$E(\sup_{s\in[0,t]} \sum_{k=1}^{\infty} (X_s[\phi_k])^2)$$

$$\leq \sum_{k=1}^{\infty} E(\sup_{s\in[0,t]} |X_s[\phi_k]|)^2 = \sum_{k=1}^{\infty} V_t(\phi_k)$$

$$\leq C_t \sum_{k=1}^{\infty} \|\phi_k\|_{p_t}^2 < \infty, \text{ where } \quad C_t = K_1^2 n_o \epsilon_t^{-2}, \text{ i.e.}$$

$$E(\sup_{s\in[0,t]} \|X_s\|_{-r_t}^2) < \infty.$$

Hence there is $G_t \in \mathcal{F}$ with $P(G_t) = 1$ such that

$$\sup_{s\in[0,t]} \|X_s(\omega)\|_{-r_t}^2 < \infty \quad \forall\, \omega \in G_t,$$

i.e. for each $\omega \in G_t$ there is a finite real number $N(\omega)$ such that

$$\sup_{s\in[0,t]} \|X_s(\omega)\|_{-r_t}^2 \leq N(\omega) < \infty.$$

Choose $q_t \geq r_t$ such that the canonical injection $i_{q_t}^{r_t} : \Phi_{q_t} \rightarrow \Phi_{r_t}$ is Hilbert–Schmidt and let $\{\psi_k : k \in \mathbb{N}\}$

be a CONS in Φ_{-q_t} consisting of elements of Φ.

By assumption on X, for each $k \in \mathbb{N}$ there is $\Omega_k \in \mathcal{F}$ with $P(\Omega_k) = 1$ such that the mapping

$$[0,t] \ni s \to X_s(\omega)[\phi_k] \text{ is CADLAG for every } \omega \in \Omega_k.$$

Let $\Omega_t = G_t \cap (\bigcap_{k=1}^{\infty} \Omega_k)$. Then $P(\Omega_t) = 1$ and

$$|X_u(\omega)[\phi_k] - X_s(\omega)[\phi_k]|^2 \leq 2N(\omega)\|\phi_k\|_{r_t}^2$$

for every $\omega \in \Omega_t$ and every $s,u \in [0,t]$. Since

$\sum_{k=1}^{\infty} \|\phi_k\|_{r_t}^2 \ \infty$ it follows by *dominated convergence* that

$$\lim_{s\downarrow u} \|X_s(\omega) - X_u(\omega)\|_{-q_t}^2 =$$

$$\lim_{s\downarrow u} \sum_{k=1}^{\infty} (X_s(\omega))[\phi_k] - X_u(\omega)[\phi_k])^2 =$$

$$\sum_{k=1}^{\infty} \lim_{s\downarrow u} (X_s(\omega)[\phi_k] - X_u(\omega)[\phi_k])^2 = 0 \quad \forall\ \omega \in \Omega_t.$$

Similarly,

$$\lim_{\substack{s\uparrow u \\ s'\uparrow u}} \|X_s(\omega) - X_{s'}(\omega)\|_{-q_t}^2 = 0 \quad \forall\ \omega \in \Omega_t.$$

Further, since $q_t \geq r_t$,

$$\sup_{s\in[0,t]} \|X_s(\omega)\|_{-q_t} \leq \sup_{s\in[0,t]} \|X_s(\omega)\|_{-r_t} < \infty$$

$\forall\ \omega \in \Omega_t$, completing the proof.

∥

REMARK 3

For reference later we note that we showed that

$E \sup_{s\in[0,t]} \|X_s\|^2_{-r_t} < \infty$. Since $q_t \geq r_t$ it follows that

$$E \sup_{s\in[0,t]} \|X_t\|^2_{-q_t} < \infty.$$

—Recall that a real-valued process $(Y_t)_{t\geq 0}$ is called progressively measurable wrt. $(\mathcal{F}_t)_{t\geq 0}$ iff

(a) Y_t is $\mathcal{F}_t$-adapted $\forall\ t\geq 0$

(b) $\forall\ t>0$: $(s,\omega) \rightarrow Y_s(\omega)$; $(s,\omega) \in [0,t]\times\Omega$ is $\mathcal{B}(\mathbb{R})/\mathcal{B}([0,t])\times\mathcal{F}_t$ measurable.

—Recall that A is a continuous linear operator on Φ and that $\{T_t : t\geq 0\}$ and A satisfy AS.1.. Let $A' : \Phi' \rightarrow \Phi'$ denote the adjoint of A.

DEFINITION

Let $M = (M_t)_{t\geq 0}$ be a Φ'-valued L^2-semimartingale and let η be a Φ'-valued random variable.

Let $\mathcal{U}_t := \{X_s, \eta : 0\leq s\leq t\} \vee \{P\text{-null sets}\}$, $t\geq 0$. A Φ'-valued process ξ_t is said to be a solution to the SDE on Φ' :

$$(14) \qquad \begin{cases} d\xi_t = A'\xi_t dt + dM_t \\ \\ \xi_o = \eta \end{cases}$$

iff

(i) $\forall\ \phi \in \Phi$: the mapping $(t,\omega) \to \xi_t(\omega)[\phi]$ is progressively measurable on $[0,\infty)\times\Omega$ wrt. $\{\mathcal{U}_t : t\geq 0\}$ and

(ii) for every $t\geq 0$:

$$(15) \qquad P\{\xi_t[\phi] = \eta[\phi] + \int_o^t \xi_u[A\phi]du + M_t[\phi]$$

$$\forall\ \phi \in \Phi\} = 1.$$

Further the equation (14) is said to have a unique solution iff for any two Φ'-valued processes $(\xi_t)_{t\geq 0}$, $(\eta_t)_{t\geq 0}$ satisfying (i) and (ii) above we have

(16) $\quad P\{\ _t = \eta_t \quad \forall\ t\geq 0\} = 1.$

III.1.5. THEOREM

Let $A : \Phi \to \Phi$ be linear and τ-continuous and suppose that A satisfies AS.1..

Let $M = (M_t)_{t\geq 0}$ be a Φ'-valued weak L^2-semimartingale and η be a Φ'-valued random variable. Then the SDE on Φ'

$$(14)\qquad \begin{cases} d\xi_t = A'\xi_t dt + dM_t \\ \xi_o = \eta \end{cases}$$

has a unique solution satisfying (10) of theorem III.1.3.. Explicitly, this solution is given by

$$\xi_t = T'_t\eta + M_t + \int_o^t A'T'_{t-s}M_s ds \quad \text{P.a.s.} \quad \forall\ t\geq 0.$$

(where T'_u denotes the adjoint of T_u considered as a continuous linear operator on Φ, and where the integral is in the weak sense).

PROOF:

By theorem III.1.4., theorem III.1.3. applies and thus the Φ'-valued process given by

(17) $\quad \xi_t = T'_t\eta + M_t + \int_0^t A'T'_{t-s}M_s ds \quad (P\text{-a.s}) \quad \forall\ t\geq 0$

is the unique (in the sense of (11) and therefore of (16) Φ'-valued process satisfying (10) and

$$P\{\xi_t[\phi] = \eta[\phi] + M_t[\phi] + \int_0^t \xi_u[A\phi]du \quad \forall\ \phi \in \Phi\} = 1$$

$\forall\ t\geq 0$.

(17) obviously implies that $\xi_t[\phi]$ is $\mathscr{G}_t$-adapted $\forall\ t\geq 0$ for every $\phi \in \Phi$, and (10) implies that $t \to \xi_t[\phi]$ is CADLAG P-a.s. for every $\phi \in \Phi$ and therefore $(t,\omega) \to \xi_t(\omega)[\phi]$ is progressively measurable wrt. $\mathscr{G}_t$ for each $\phi \in \Phi$ (by Meyer; [20] theorem ^{m} 47).

Hence ξ_t given by (17) is the unique solution to (14) satisfying (10).

//

III.1.6. PROPOSITION

Let $M = (M_t)_{t\geq 0}$ be a Φ'-valued weak L^2-semimartingale, and let η be a Φ'-valued random variable satisfying

$E\|\eta\|^2_{-r} < \infty$ for some $r \in \mathbb{N}_0$. If either

(i) $\quad \eta \perp\!\!\!\perp \{M_s : s\geq 0\}$ or

(ii) η is $\mathbf{F}_o$-measurable

then the Φ'-valued process

$$\xi_t = T'_t\eta + M_t + \int_o^t A'T'_{t-s}M_s ds \quad (P\text{-a.s.})$$

is a Φ'-valued weak L^2-semimartingale wrt. $(\mathcal{G}_t)_{t>0}$.

PROOF:

We already know that $\xi_t[\phi]$ *is* $\mathcal{G}_t$*-adapted for every* $\phi \in \Phi$ and if either (i) or (ii) holds then $(M_t[\phi])_{t\geq 0}$ is a $(\mathcal{G}_t)_{t\geq 0}$-$L^2$-semimartingale for every $\phi \in \Phi$. Therefore it suffices to show that for each $\phi \in \Phi$ the process

$$(\eta[T_t\phi] + \int_o^t M_s[T_{t-s}A\phi]ds)_{t\geq 0}$$

is a CADLAG L^2-process of bounded variation on compact sets. But it follows from lemma III.1.1. that $t \to \eta(\omega)[T_t\phi]$ is differentiable for each $\phi \in \Phi$ and each $\omega \in \Omega$, and the mapping

$$t \to \int_o^t M_s(\omega)[T_{t-s}A\phi]ds$$

is absolutely continuous for P-a.s. $\omega \in \Omega$. Thus it only remains to show that for every $\phi \in \Phi$

$$E(\eta[_t\phi] + \int_o^t M_s[T_t-_sA\phi]ds)^2 < \infty \quad \forall\ t\geq 0 :$$

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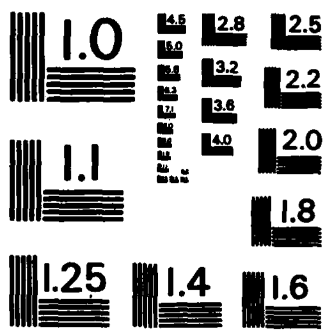

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Fix $\phi \in \Phi$ and $t \geq 0$.

By Remark 3 there is $r_t \in \mathbb{N}_o$ such that

$$E \sup_{0 \leq s \leq t} \|M_s\|^2_{-r_t} < \infty. \text{ Hence}$$

$$E(\eta[T_t\phi] + \int_o^t M_s[T_{t-s}A\phi]ds)^2 \leq$$

$$2E(\eta[T_t\phi])^2 + 2E(\int_o^t M_s[T_{t-s}A\phi]ds)^2 \leq$$

$$2(E\|\eta\|^2_{-r})\|T_t\phi\|^2_r +$$

$$2E\int_o^t \int_o^t \|M_s\|_{-r_t}\|M_u\|_{-r_t}\|T_{t-s}A\phi\|_{r_t}\|T_{t-u}A\phi\|_{r_t}dsdu$$

$$\leq 2\|T_t\phi\|^2_r E\|\eta\|^2_{-r} +$$

$$2\int_o^t \int_o^t (E\|M_s\|^2_{-r_t}E\|M_u\|^2_{-r_t})^{1/2}\|T_{t-s}A\phi\|_{r_t}\|T_{t-u}A\phi\|_{r_t}dsdu$$

$$\leq 2\|T_t\phi\|^2_r E\|\eta\|^2_{-r} +$$

$$2E(\sup_{0 \leq s \leq t} \|M_s\|^2_{-r_t})(\int_o^t \|T_{t-s}A\phi\|_{r_t}ds)^2 < \infty,$$

since $E\|\eta\|^2_{-r} < \infty$ by assumption and $s \to \|T_{t-s}A\phi\|_{r_t}$ is continuous on $[0,t]$ by AS.1. (c).

For any $T>0$ and $q \in \mathbb{N}_o$ let $D([0,T],\Phi_{-q})$ denote the Skorohod space of all Φ_{-q}-valued functions F on $[0,T]$

which are CADLAG wrt. $\|\cdot\|_{-q}$. $D([0,T],\Phi_{-q})$ is a complete separable metric space under the metric constructed by Lindvall [19] (see also [14]).

III.1.7. COROLLARY

Let $M = (M_t)_{t\geq 0}$ be a Φ'-valued weak semimartingale and let η be a Φ'-valued random variable satisfying either (i) or (ii) of proposition III.1.6. and $E\|\eta\|^2_{-r}$ for some $r \in \mathbb{N}_o$.

Let ξ_t denote the unique solution to (14) satisfying (10) whose existence was shown in theorem III.1.5..

Then, for every T>0, there exists $\Omega_T \in \mathcal{F}$ with $P(\Omega_T) = 1$ and $p_T \in \mathbb{N}_o$ such that

$$\xi^T(\omega) := (\xi_t(\omega))_{t\in[0,T]} \in D([0,T],\Phi_{-p_T}) \quad \forall \omega \in \Omega_T.$$

PROOF:

ξ_t is given by

$$\xi_t = T'_t\eta + M_t + \int_0^t A'T'_{t-s}M_s ds \quad (P\text{-a.s.})$$

and therefore $(\xi_t)_{t\geq 0}$ is a Φ'-valued weak L^2-semimartingale by proposition III.1.6.. Hence ξ_t satisfies AS.2. by theorem III.1.4.. But AS.2. is equivalent to the assertion of the corollary.

$\#$

For any Φ'-valued process $Y = (Y_t)_{t\geq 0}$ define for $T>0$
$Y^T := (Y_t)_{t\in[0,T]}$.

If $M = (M_t)_{t\geq 0}$ is a Φ'-valued weak semimartingale then, by theorem III.1.4., for every $T>0$ there exists $q_T \in \mathbb{N}_0$ such that

$$M^T \in D([0,T],\Phi_{-q_T}) \quad P\text{-a.s.}.$$

Corollary III.1.7. says that for any "reasonable" initial condition η there is $p_T \in \mathbb{N}_0$ such that

$$\xi^T \in D([0,T],\Phi_{-p_T}) \quad P\text{-a.s.}.$$

If $q_T = \min\{q \in \mathbb{N}_0 : M^T \in D([0,T],\Phi_{-q})\}$

then, it is clear from the expression for ξ_t that in general $p_T \geq q_T$. However, when Φ and the operator A have special properties frequently encountered in praxis we may always take $p_T = q_T$, provided that $\eta \in \Phi_{-q_T}$. To see this, we first prepare some auxiliary results:

III.1.8. LEMMA

Every uncountable analytic space is of the second category.

<u>PROOF:</u>

Let S be an uncountable analytic space. Then S contains a subset K which is homeomorphic to the irrationals (see J. Hoffman-Jørgensen & F. Topsøe [7], theorem 7 page 22 and subsequent remarks). Since the irrationals are of the second category, so is K and hence S.

<u>III.1.9. LEMMA</u>

If $\Phi \neq \{0\}$, then $(\Phi, \|\cdot\|_p)$ is of the second category for each $p \in \mathbb{N}_o$.

<u>PROOF:</u>

Let $p \in \mathbb{N}_o$ and let $\dot{\iota}_p : \Phi \to \Phi_p$ denote the canonical injection. Then, since $\dot{\iota}_p$ is continuous, $(\Phi, \|\cdot\|_p)$ is a continuous image of the Polish (i.e. complete separable metric) space (Φ, τ), and thus $(\Phi, \|\cdot\|_p)$ is analytic, so the conclusion follows from lemma III.1.8., since every real vector space of dimension ≥ 1 is uncountable.

For $p \in \mathbb{N}_o$ let $\langle.,.\rangle_p$ denote the inner product on Φ_p,

i.e. $$\langle\phi,\psi\rangle_p = \frac{1}{2}(\|\phi + \psi\|_p^2 - \|\phi\|_p^2 - \|\psi\|_p^2).$$

DEFINITION

A set $\{\phi_j : j \in \mathbb{N}\} \subset \Phi$ such that

(i) $\{\phi_j : j \in \mathbb{N}\}$ is a CONS in Φ_o and

(ii) $\forall\ p \in \mathbb{N}_o\ \forall\ \phi,\psi \in \Phi_p$:

$$<\phi,\psi>_p = \sum_{j=1}^{\infty} <\phi,\phi_j>_o<\psi,\phi_j>_o\|\phi_j\|_p^2$$

is called a common orthogonal system for $\{\Phi_p : p \in \mathbb{N}_o\}$.

III.1.10. LEMMA

Suppose that Φ has a common orthogonal system $\{\phi_j : j \in \mathbb{N}\}$ for $\{\Phi_p : p \in \mathbb{N}_o\}$.

If $B : \Phi_o \to \Phi_o$ is a bounded linear operator satisfying $B\Phi \subset \Phi$, then

(18) $B\Phi_p \subset \Phi_p \quad \forall\ p \in \mathbb{N}_o$

(19) $B|_{\Phi_p}$ is $\|\cdot\|_p$-continuous $\quad \forall\ p \in \mathbb{N}_o$.

PROOF:

Let $p \in \mathbb{N}_o$. For each $k \in \mathbb{N}$, define

$$f_p^k(\phi) = \sum_{j=1}^{k} \langle B\phi, \phi_j \rangle_o^2 \|\phi_j\|_p^2; \; \phi \in \Phi$$

Since $\|\phi\|_p \geq \|\phi\|_o \; \forall \; p \in \mathbb{N}_o, \; \forall \; \phi \in \Phi$ and since B is continuous on Φ_o, f_p^k is a $\|\cdot\|_p$-continuous function of $\phi \in \Phi$ for each k.

Also, $\sup_{k\in\mathbb{N}} f_p^k(\phi) = \|B\phi\|_p^2 < \infty \quad \forall \; \phi \in \Phi$ since $B\Phi \subset \Phi$.

Therefore,

$$f_p(\phi) := \sup_{k\in\mathbb{N}} f_p^k(\phi)$$

is a lower semicontinuous function on $(\Phi, \|\cdot\|_p)$. Moreover

$f_p(a\phi) = |a|^2 f_p(\phi) \quad \forall \; a \in \mathbb{R}$ and f_p is convex on Φ.

Hence, for any $n \in \mathbb{N}$, $A_n = \{\phi \in \Phi : f_p(\phi) \leq n\}$ is a closed, convex and balanced subset of $(\Phi, \|\cdot\|_p)$. Further,

$$\Phi = \bigcup_{n\geq 1} A_n.$$

Since $\{\phi_j : j \in \mathbb{N}\}$ is a common orthogonal system for $\{\Phi_p : p \in \mathbb{N}_o\}$, $\Phi \neq \{0\}$ and thus lemma III.1.9. implies the existence of $n_o \in \mathbb{N}$ such that

$$\text{interior}(A_{n_o}) \neq \emptyset \text{ in } (\Phi, \|\cdot\|_p).$$

Since A_{n_o} is convex, balanced and contains zero, A_{n_o} contains a zero-neighbourhood in $(\Phi, \|\cdot\|_p)$; i.e. there is $\epsilon > 0$ such that

$$\{\phi \in \Phi : \|\phi\|_p < \epsilon\} \subset \{\phi \in \Phi : f_p(\phi) \leq n_o\}$$

But then there is a constant K such that

$$f_p(\phi) \leq K\|\phi\|_p^2 \quad \forall \phi \in \Phi; \quad \text{i.e.}$$

$$\|B\phi\|_p^2 \leq K\|\phi\|_p^2 \quad \forall \phi \in \Phi.$$

Since Φ is dense in Φ_p (18) and (19) follow.

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III.1.11. THEOREM

Let $H = \Phi_o$ and suppose that Φ has a common orthogonal system $\{\phi_j : j \in \mathbb{N}\}$ for $\{\Phi_p : p \in \mathbb{N}_o\}$. Suppose further that (in addition to satisfying AS.1.) A is dissipative and selfadjoint on $H = \Phi_o$.

Let $M = (M_t)_{t\geq 0}$ be a Φ'-valued weak L^2-semimartingale and suppose that $M^T \in D([0,T], \Phi_{-q_T})$ (P-a.s.) for some T>0.

Let η be a Φ'-valued random variable such $E\|\eta\|_{-r}^2 < \infty$ for some r>0 and suppose that η satisfies either (i) or (ii) of proposition III.1.6.. If $\eta(\omega) \in \Phi_{-q_T}$ $\forall \omega \in \Omega$, then

$\xi^T \in D([0,T],\Phi_{-q_T})$ (P-a.s.) where

$$\xi_t = T_t'\eta + \int_0^t A'T_{t-s}'M_s ds + M_t.$$

PROOF:

Since T_t is a bounded linear operator on $H = \Phi_0$ and $T_t\Phi \subset \Phi$ by AS.1. (a), lemma III.1.10. gives $T_t\Phi_{q_T} \subset T_t\Phi_{q_T}$ $\forall$ $t \geq 0$. Hence $T_t'\Phi_{-q_T} \subset T_t'\Phi_{-q_T}$ $\forall$ $t \geq 0$ and therefore

$\|T_t'\eta\|_{-q_T} < \infty$ $\forall$ $t \geq 0$. Also,

$\|M_t\|_{-q_T} < \infty$ $\forall$ $t \in [0,T]$ P-a.s. by assumption.

To show that also

$\|\int_0^t A'T_{t-s}'M_s ds\|_{-q_T} < \infty$ $\forall$ $t \in [0,T]$ (P-a.s.)

it sufficies to show that $\phi \to \int_0^t M_s(\omega)[T_{t-s}A\phi]ds$

extends to a continuous linear functional on Φ_{q_T} for P-a.s. $\omega \in \Omega$:

Since A is selfadjoint on Φ_0 so is T_t for each $t \geq 0$ and since $T_tA = AT_t$ $\forall$ $t \geq 0$, T_tA is selfadjoint. By the spectral theorem we therefore have

$$\langle T_t A\phi, \phi\rangle_o = \int_{\sigma(A)} \lambda e^{\lambda t} d\langle E(\lambda)\phi, \phi\rangle_o \quad \forall\ \phi \in \Phi$$

where $\sigma(A)$ = Spectrum(A) and $E(\lambda)$ is the unique resolution of the identity on Φ_o associated with A. Since A is dissipative on Φ_o, $\sigma(A) \subset (-\infty, 0]$ and hence

$$|\langle T_t A\phi, \phi\rangle_o| \leq K_t \|\phi\|_o^2 \quad \forall\ t>0 \quad \forall\ \phi \in \Phi$$

Where $K_t = \sup_{\lambda \in (A)} |\lambda e^{\lambda t}| \leq \frac{1}{et} < \infty \quad \forall\ t>0.$

Since Φ is dense in Φ_o, $T_t A$ extends to a continuous linear operator on Φ_o for each $t>0$. By AS.1. we also have $T_t A\Phi \subset \Phi$ so lemma III.1.10. gives

$T_t A\Phi_{q_T} \subset \Phi_{q_T}$ and

(20) $\quad T_t A|_{\Phi_{q_T}}$ is $\|\cdot\|_{q_T}$ continuous $\quad \forall\ t>0.$

By assumption there is $\Omega_T \in \mathcal{F}$ with $P(\Omega_T) = 1$ such that

$t \to M_t(\omega)$ is $\|\cdot\|_{-q_T}$–CADLAG on $[0,T]$ for each $\omega \in \Omega_T$. But then

(21) $\quad \int_o^t |M_s(\omega)[T_{t-s}A\phi]|ds < \infty \quad \forall\ \phi \in \Phi\ \forall\ t \in [0,T]$
$\forall\ \omega \in \Omega_T$, because

(22) $\quad |M_s(\omega)[T_{t-s}A\phi]| \leq \|M_s(\omega)\|_{-q_T} \|T_{t-s}A\phi\|_{q_T}$

and since $\phi \in \Phi$, $s \to \|T_{t-s}A\phi\|_{q_T}$ is continuous on $[0,T]$ by AS.1 (c). Let $n \in \mathbb{N}$. We claim that:

(23) $$\phi \to \int_0^{t-1/n} |M_s(\omega)[T_{t-s}A\phi]|ds \text{ is continuous on } (\Phi, \|\cdot\|_{q_T}) \text{ for all } \omega \in \Omega_T:$$

Let $\phi, \psi_k \in \Phi$ and suppose that $\|\phi - \psi_k\|_{q_T} \to 0$ as $k \to \infty$.

By (20) $\|T_{t-s}A[\phi-\psi_k]\|_{q_T} \xrightarrow[k\to\infty]{} 0$ for each $s \in [0,t-1/n]$. Hence

$$f_n(s) := \sup_{k\in\mathbb{N}} \|T_{t-s}A(\phi-\psi_k)\|_{q_T} < \infty \quad \forall\ s \in [0,t-1/n]$$

and since $s \to \|T_{t-s}(\phi-\psi_k)\|_{q_T}$ is continuous on $[0,t-1/n]$, $f_n(s)$ is lower semicontinuous on $[0,t-1/n]$, and thus $f_n \in L^1([0,t-1/n])$. Therefore

$$\int_0^{t-1/n} |M_s(\omega)[T_{t-s}A(\phi-\psi_k)]|ds \xrightarrow[k\to\infty]{} 0 \ \forall \ \omega \in \Omega_T$$

by (22) and dominated convergence.

Define, for $t \in [0,T]$ and $\omega \in \Omega_T$ fixed,

$$g_{t,\omega}(\phi) = \sup_n \int_0^{t-1/n} |M_s(\omega)[T_{t-s}A\phi]|ds; \ \phi \in \Phi$$

Then, by (21) and (23) $g_{t,\omega}$ is a lower semicontinuous function on $(\Phi, \|\cdot\|_{q_T})$. Moreover,

$g_{t,\omega}(a\phi) = |a| g_{t,\omega}(\phi) \quad \forall\ a \in \mathbb{R}$ and

$g_{t,\omega}$ is convex on Φ.

Since $(\Phi, \|\cdot\|_{q_T})$ is of the second category by lemma III.1.9., it follows by a now familiar argument that there is a constant $C(t,\omega)$ such that

$$g_{t,\omega}(\phi) \leq C(t,\omega) \|\phi\|_{q_T} \quad \forall\ \phi \in \Phi.$$

Hence, for each $t \in [0,T]$ and $\omega \in \Omega_T$, $g_{t,\omega}$ extends to a continuous function on Φ_{q_T}. But then

$$\left| \int_0^t M_s(\omega)[T_{t-s}A\phi]ds - \int_0^t M_s(\omega)[T_{t-s}A\psi]ds \right| \leq$$

$$g_{t,\omega}(\phi - \psi) \leq C(t,\omega) \|\phi - \psi\|_{q_T} \text{ and thus}$$

$$\phi \to \int_0^t M_s(\omega)[T_{t-s}A\phi]ds$$

is $\|\cdot\|_{q_T}$-continuous on Φ for each $t \in [0,T]$ and $\omega \in \Omega_T$. Since Φ is dense in Φ_{q_T},

$$\phi \to \int_0^t M_s(\omega)[T_{t-s}A\phi]ds \quad \text{is continuous on } \Phi_{q_T}, \text{ i.e.}$$

$$\int_0^t A'T'_{t-s}M_s(\ )ds \in \Phi_{-q_T} \quad \forall\ t \in [0,T],\ \forall\ \omega \in \Omega_T.$$

Hence $\xi_t(\omega) \in \Phi_{-q_T}\ \forall\ t \in [0,T]\ \forall\ \omega \in \Omega_T$.

To show that $t \to \xi_t$ is $\|\cdot\|_{-q_T}$-CADLAG on $[0,T]$ (P-a.s.), we note that the conditions of Corollary III.1.7. are satisfied, and thus there is $p_T \in \mathbb{N}$ and $G_T \in \mathcal{F}$ with $P(G_T) = 1$ such that

$$(24) \qquad \xi^T(\omega) \in D([0,T],\Phi_{-p_T}) \quad \forall\, \omega \in G_T.$$

Fix $s \in [0,T]$. Let $t_n \downarrow s$ as $n \to \infty$. Then by (24)

$$\xi_{t_n}(\omega)[\phi] \to \xi_s(\omega)[\phi] \quad \forall\, \phi \in \Phi \quad \forall\, \omega \in G_T,$$

i.e. for every $\omega \in G_T$

$$\xi_{t_n}(\omega) \xrightarrow[n\to\infty]{} \xi_s(\omega) \text{ weakly on } \Phi'.$$

Since Φ is countably Hilbert nuclear this implies that

$$\xi_{t_n}(\omega) \xrightarrow[n\to\infty]{} \xi_s(\omega) \text{ strongly on } \Phi'.$$

Since $\xi_t(\omega) \in \Phi_{-q_T}$ $\forall\, 0 \le t \le T$ $\forall\, \omega \in \Omega_T$ and since (Φ', σ) is the <u>strict</u> inductive limit of

$\{\Phi_{-q} : q \in \mathbb{N}\}$ this means that

$$\|\xi_{t_n}(\omega) - \xi_s(\omega)\|_{-q_T} \to 0 \;\; \forall\, \omega \in G_T \cap \Omega_T$$

But $(\Phi_{-q_T}; \|\cdot\|_{-q_T})$ is a metric space, so sequential right continuity wrt. $\|\cdot\|_{-q_T}$ implies right continuity wrt. $\|\cdot\|_{-q_T}$. Therefore,

$\xi_t(\omega)$ is $\|\cdot\|_{-q_T}$-right continuous at $s \in [0,T)$ for every $\omega \in \Omega_T \quad G_T$.

In a similar fashion we show that the left limit $\xi_{s-}(\omega)$ exists in $\|\cdot\|_{-q_T}$ for $s \in (0,T]$ for every $\omega \in G_T \cap \Omega_T$.

Hence $\xi^T(\omega) \in D([0,T];\Phi_{-q_T})$ $\forall$ $\omega \in G_T \cap \Omega_T$ completing the proof.

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REMARK 4

Corollary III.1.7. may be derived without assuming that η satisfies either (i) or (ii) of proposition III.1.6., but the proof is rather long and tedious and since the resulting gain in generality is practically insignificant we omit it. Instead we note that this assumption may consequently also be dropped from theorem III.1.11.

REMARK 5

The class of countably Hilbert nuclear spaces possessing a common orthogonal system for the generating sequence of Hilbert spaces $\{\Phi_p : p \in \mathbb{N}_o\}$ is rather large and in particular it contains any nuclear space generated in the manner discussed in Chapter II and Appendix. In particular it contains the Schwartz space $\mathscr{S}(\mathbb{R}^d)$ of all rapidly decreasing functions on $\mathbb{R}^d$.

—Theorem III.1.11 is a generalization of a very recent result by R. T. Chari ([4], 1985). He shows the following: ($\bar{A}$ denotes the closure of A in H)

Let $\Phi = \mathscr{S}(\mathbb{R}^d)$; $H = \Phi_o = L^2(\mathbb{R}^d)$. Let $A : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ be continuous, linear and suppose that $\bar{A}$ is selfadjoint and dissipative on H. Suppose that there is a strongly continuous semigroup of bounded linear operators $\{T_t : t \geq 0\}$ on $\mathscr{S}(\mathbb{R}^d)$ satisfying

(25) $\quad F[T_t\phi] - F[\phi] =$

$$\int_o^t F[AT_s\phi]ds \quad t \geq 0 \quad \forall F \in \mathscr{S}'(\mathbb{R}^d) \ \forall \phi \in \mathscr{S}(\mathbb{R}^d).$$

(in view of his other assumptions on A and $\{T_t : t \geq 0\}$ (25) amounts to saying that A is the strong generator of $\{T_t : t \geq 0\}$ in the $\mathscr{S}(\mathbb{R}^d)$-topology)

Let $M = (M_t)_{t \geq 0}$ be an $\mathscr{S}'(\mathbb{R}^d)$-valued weak martingale for which there exists $q > 1$ such that

$$\forall\, T>0 : M^T \in D([0,T],\Phi_{-q}) \text{ P-a.s.}$$

If $E\|\eta\|^2_{-q} < \infty$, then the SDE on $\mathscr{S}'(\mathbb{R}^d)$

$$\begin{cases} d\xi_t = A'\xi_t dt + dM_t \\ \xi_0 = \eta \end{cases}$$

has a unique solution $\xi = (\xi_t)_{t\geq 0}$ satisfying

$\forall\ T>0 : \xi^T \in D([0,T],\Phi_{-q})$ P-a.s..

As Chari remarks ([4], page 10): "It is easily checked that $T_t f = e^{t\bar{A}} f$ for $f \in \mathscr{S}(\mathbb{R}^d)$, (where $e^{t\bar{A}}$ is the semigroup of selfadjoint contractions on $L^2(\mathbb{R}^d)$ generated by $\bar{A}$)". This in combination with (25) implies that our assumption AS.1. is satisfied in Chari's case and in view of Remarks 4 and 5 above we therefore see that Chari's result is a special case of theorem III.1.11. and theorem III.1.5..

His method of proof is quite different from ours, however, and makes use of finite dimensional approximations, obtained through a theorem by Doleans-Dade ([5]), to the solution and then it relies heavily upon the existence of a common orthogonal system in $\mathscr{S}(\mathbb{R}^d)$ as well as the dissipativity of A. Although his method gives that $\xi^T \in D([0,T],\Phi_{-q})$ rather painlessly, it does not provide an explicit formula for the solution. Also, as theorem III.1.5. shows, neither dissipativity of A nor the existence of a common orthogonal system are essential for the existence and uniqueness part. Further, theorem

III.1.11 shows that the assumption that the noise "lives" in the same Φ_{-q} for all $t \geq 0$ is not material to the conclusion. In fact, this assumption makes the nuclear structure of $\mathcal{S}$ superfluous (the fact that $\mathcal{S}$ is nuclear does not enter Chari's proof at all) and in effect reduces the problem to solving SDE's on a Hilbert space.

REMARK 6

If the Φ'-valued weak L^2-semimartingale $M = (M_t)_{t \geq 0}$ has the property that

(26) $\quad \forall\, \phi \in \Phi : (M_t[\phi])_{t \geq 0}$ is a continuous real L^2-semimartingale (P-a.s.)

then the spaces $D([0,T],\Phi_{-p_T})$ and $D([0,T],\Phi_{-q_T})$ in respectively Corollary III.1.7. and theorem III.1.11. may be replaced by the spaces $C([0,T],\Phi_{-p_T})$, respectively $C([0,T],\Phi_{-q_T})$; where $C[[0,T],\Phi_{-r})$ denotes the complete metric space of all $\|\cdot\|_{-r}$-continuous functions $f : [0,T] \to \Phi_{-r}$.

Notice that when (26) holds then

(27) $\quad \forall\, T>0 \; \exists r_T \in \mathbb{N}_0 : M^T \in C([0,T];\Phi_{-r_T}) \quad$ P-a.s.

((27) may be proved following the exact same procedure as was used in the proof of theorem III.1.4.). The necessary changes in the proofs of Corollary III.1.7. respectively theorem III.1.11. are obvious and therefore omitted.

Hitherto we have not been concerned with the construction of Φ'-valued weak L^2-semimartingales. In fact, our definition of these presupposes that a Φ'-valued process $M = (M_t)_{t\geq 0}$ is already given and then it is an L^2-semimartingale wrt. a filtration $(\mathbb{F}_t)_{t\geq 0}$ if $M_t[\phi]$ is a real L^2-semimartingale wrt. $\mathbb{F}_t$ for each $\phi \in \Phi$. In praxis, however, one is often given a family $\{\bar{M}(\phi) : \phi \in \Phi\}$ such that $\bar{M}(\phi) = (\bar{M}_t(\phi))_{t\geq 0}$ is a real semimartingale for each $\phi \in \Phi$ and such that

$$\bar{M}_t(\lambda_1\phi_1 + \lambda_2\phi_2) = \lambda_1\bar{M}_t(\phi_1) + \lambda_2\bar{M}_t(\phi_2) \quad \text{P-a.s.}$$

for each $t\geq 0$, $(\lambda_1,\phi_1,\lambda_2,\phi_2) \in \mathbb{R}\times\Phi\times\mathbb{R}\times\Phi$,

and so the question is whether there exists a Φ'-valued process $M = (M_t)_{t\geq 0}$ such that

$$M_t[\phi] = \bar{M}_t[\phi] \quad \forall\ t\geq 0 \quad \text{P-a.s.} \quad \forall\ \phi \in \Phi.$$

The following result, which uses a technique devised by K. Itô in [12] known as regularization, gives a sufficient condition which is often useful:

III.1.12 THEOREM

Let $\bar{M}_t = \{\bar{M}_t(\phi) : \phi \in \Phi\}$; $t\geq 0$ be a family of real valued stochastic processes. If $(\bar{M}_t)_{t\geq 0}$ has the properties

(L) $\quad \bar{M}_t(c_1\phi_1 + c_2\phi_2) = c_1\bar{M}_t(\phi_1) + c_2\bar{M}_t(\phi_2) \quad$ P.a.s.

$\forall\ t\geq 0$, $\quad (c_1,c_2,\phi_1,\phi_2) \in \mathbb{R}\times\mathbb{R}\times\Phi\times\Phi$ (note that the exceptional ω-set may depend upon the choice of (c_1,c_2,ϕ_1,ϕ_2) and t)
and

(B) $\quad \forall\ T>0\ \exists C_T > 0\ \exists r_T \in \mathbb{N}_o : E\sup_{0\leq t\leq T} (\bar{M}_t(\phi))^2 \leq C_T\|\phi\|^2_{r_T}$
$\forall\ \phi \in \Phi$, and

(C) $\quad \forall\ \phi \in \Phi : (\bar{M}_t(\phi])_{t\geq 0}$ is an L^2-semimartingale.

Then

(a): There exists a Φ'-valued weak L^2-semimartingale $M = (M_t)_{t\geq 0}$ such that

$$M_t[\phi] = \bar{M}_t(\phi) \quad P\text{-a.s.} \quad \forall\ \phi \in \Phi,\ \forall\ t\geq 0.$$

(b): If $\{\bar{M}_t(\phi) : \phi \in \Phi\}$; $t\geq 0$ satisfies (L), (C) and

(B') $\quad \exists r \in \mathbb{N}_o\ \forall\ T>0\ \exists\ C_T>0 : E\sup_{0\leq t\leq T} (\bar{M}_t(\phi))^2 \leq C_T\|\phi\|^2_r$
$\forall\ \phi \in \Phi$,

then there is $q \in \mathbb{N}_o$ and a Φ_{-q}-valued CADLAG process $M = (M_t)_{t\geq 0}$ such that

$$M_t[\phi] = \bar{M}_t(\phi) \quad \forall\ t\geq 0\ (P\text{-a.s.})\ \forall\ \phi \in \Phi$$

(and consequently M is also a Φ'-valued weak L^2-semimartingale).

<u>PROOF:</u>

(a): Let $T_o = 0$, $T_n > T_{n-1}$; $n\geq 1$ with $T_n \uparrow \infty$. By (B), for each $n \in \mathbb{N}$, there is $r_n \in \mathbb{N}_o$ such that

$$E \sup_{0\leq t\leq T_n} |\bar{M}_t(\phi)|^2 \leq C_{T_n} \|\phi\|_{r_n}^2 \qquad \forall\ \phi \in \Phi$$

For each $n \in \mathbb{N}$ choose q_n such that the canonical injection $i_{q_n}^{r_n}$ is Hilbert-Schmidt. Let $\{\phi_k^n : k \in \mathbb{N}\}$ be a CONS in Φ_{q_n} consisting of elements of Φ, and let $\{f_k^n : k \in \mathbb{N}\}$ be the CONS in Φ_{-q_n} dual to $\{f_k^n : n \in \mathbb{N}\}$ (i.e. $f_k^n[\phi_j^n] = \delta_{kj}\ \forall\ n$).

By (B) and Hilbert-Schmidtness of $i_{q_n}^{r_n}$ we have for each $n \in \mathbb{N}$:

$$E \sup_{0\leq t\leq T_n} \sum_{k=1}^{\infty} (\bar{M}_t(\phi_k^n))^2 \leq \sum_{k=1}^{\infty} E \sup_{0\leq t\leq T_n} (\bar{M}_t(\phi_k^n))^2$$

$$\leq C_{T_n} \sum_{k=1}^{\infty} \|\phi_k^n\|_{r_n}^2 < \infty.$$

Hence for each $n \in \mathbb{N}$ there is $\Omega_n \in \mathcal{F}$ with $P(\Omega_n) = 1$ such that

$$\sup_{0 \leq t \leq T_n} \sum_{k=1}^{\infty} (\bar{M}_t(\phi_k^n,\omega))^2 < \infty \quad \forall\ \omega \in \Omega_n.$$

Put $G = \bigcap_{n \geq 1} \Omega_n$. Then $P(G) = 1$ and

$$\forall\ n \in \mathbb{N} : \sup_{0 \leq t \leq T_n} \sum_{k=1}^{\infty} (\bar{M}_t(\phi_k^n,\omega))^2 < \infty \quad \forall\ \omega \in G,$$

$\forall\ t \in [0,T_n]$.

Define

$$M_t^n(\omega) = \begin{cases} \sum_{k=1}^{\infty} \bar{M}_t(\phi_k^n,\omega) f_k^n & \text{if } \omega \in G \\ 0 & \text{if } \omega \notin G \end{cases} ; \quad t \in [0,T_n]$$

Then, for each $n \in \mathbb{N}$ we have

$$M_t^n(\omega) \in \Phi_{-q_n} \quad \forall\ \omega \in G,\ \forall\ t \in [0,T_n].$$

and $\Omega \ni \omega \to M_t^n(\omega)$ is a Φ_{-q_n}-valued random variable for each $t \in [0,T_n]$.

Define

$$M_t(\omega) = M_t^1(\omega)1_{[0,T_1]} + \sum_{n=2}^{\infty} M_t^n(\omega)1_{(T_{n-1},T_n]}(t)$$

then M_t is a Φ'-valued random variable for every $t\geq 0$.

Fix $t>0$. Then there is $n \in \mathbb{N}$ such that $t \in (T_{n-1}, T_n]$. But then

$$M_t[\phi_k^n] = M_t^n[\phi_k^n] = \bar{M}_t(\phi_k^n) \quad \text{P.a.s.}$$

for each $k \in \mathbb{N}$. Hence by (L)

$$M_t[\phi] = \bar{M}_t(\phi) \quad \text{P.a.s.} \quad \forall\ \phi \in \text{span}\{\phi_k^n : k \in \mathbb{N}\}$$

But span $\{\phi_k^n \cdot k \in \mathbb{N}\}$ is dense in Φ_{r_n} and (L) and (B) imply that $\tilde{M}_t$ extends to a bounded linear operator from Φ_{r_n} into $L^2(\Omega,\mathcal{F},P)$. Hence it follows by continuity and the fact that $\|\phi - \phi_n\|_{q_n} \to 0 \Rightarrow \|\phi - \phi_n\|_{r_n} \to 0$ that

$$M_t[\phi] = \bar{M}_t(\phi) \quad \text{P-a.s.} \quad \forall\ \phi \in \Phi_{q_n}, \quad \text{in particular}$$
$\forall\ \phi \in \Phi$.

If $t=0$, then a similar argument gives

$$M_o[\phi] = \bar{M}_o(\phi) \quad \text{P-a.s.} \quad \forall\ \phi \in \Phi. \text{ Hence}$$

$$M_t[\phi] = \bar{M}_t(\phi) \quad \text{P-a.s.} \quad \forall\ \phi \in \Phi,\ \forall\ t\geq 0,$$

and therefore (C) implies that $M = (M_t)_{t\geq 0}$ is a Φ'-valued weak L^2-semimartingale. This concludes the proof of (a).

(b): Since Φ is a countably Hilbert nuclear space there is $p \in \mathbb{N}$ such that the canonical injection $i_p^r : \Phi_p \to \Phi_r$ is Hilbert–Schmidt. Let $q = \min\{p \geq r : i_p^r$ is Hilbert–Schmidt$\}$. Let $\{\phi_k : k \in \mathbb{N}\}$ be a CONS in Φ_q consisting of elements of Φ and let $\{f_k : k \in \mathbb{N}\}$ be the CONS in Φ_{-q} dual to $\{\phi_k : k \in \mathbb{N}\}$

(i.e. $f_k[\phi_j] = \delta_{kj} \quad \forall\ k,j \in \mathbb{N}$).

By (B′) and Hilbert–Schmidtness of i_q^r we have for each T>0

$$E \sup_{0\leq t\leq T} \sum_{k=1}^{\infty} (M_t(\phi_k))^2 \leq \sum_{k=1}^{\infty} E \sup_{0\leq t\leq T} (\bar{M}_t(\phi_k))^2$$

$$\leq C_T \sum_{k=1}^{\infty} \|\phi_k\|_r^2 < \infty.$$

Hence, for each t>0, there is $\Omega_T \in \mathcal{F}$ with $P(\Omega_T) = 1$ such that

$$\sup_{0\leq t\leq T} \sum_{k=1}^{\infty} (\bar{M}_t(\phi_k,\omega))^2 < \infty \qquad \forall\ \omega \in \Omega_T.$$

Let $T_n \uparrow \infty$ and put $G_1 = \bigcap_{n\geq 1} \Omega_{T_n}$. Then $P(G_1) = 1$ and

$$\forall\ \omega \in G_1 : \sup_{0\leq t\leq T} \sum_{k=1}^{\infty} (\bar{M}_t(\omega)(\phi_k))^2 < \infty \qquad \forall\ T>0.$$

By (C) for each $k \in \mathbb{N}$ there is $B_k \in \mathcal{F}$ with $P(B_k) = 1$ such that

$t \to \bar{M}_t(\omega)(\phi_k)$ is CADLAG $\quad \forall\ \omega \in B_k$.

Let $G = G_1 \cap (\bigcap_{k\geq 1} B_k)$. Then $P(G) = 1$. Define

$$M_t(\ ) = \begin{cases} \sum_{k=1}^{\infty} \bar{M}_t(\omega)(\phi_k) f_k & \text{if } \omega \in G \\ 0 & \text{if } \omega \notin G \end{cases} \quad ; t \geq 0$$

Then $M_t(\omega) \in \bar{\Phi}_{-q}\ \forall\ \omega \in G\ \forall\ T \geq 0$.

Fix $T>0$ and for $n \in \mathbb{N}$ put

$$f_t^n(\omega) = \begin{cases} \sum_{k=1}^{n} \bar{M}_t(\phi_k,\omega) f_k & \text{if } \omega \in G \\ 0 & \text{if } \omega \notin G \end{cases} \quad ; t \in [0,T]$$

By definition of G the mapping

$t \to f_t^n(\omega)$ is CADLAG on $[0,T]$ wrt. $\|\cdot\|_{-q}$ for <u>every</u> $\omega \in \Omega$, i.e.

$$f^n(\omega) \in D([0,T],\bar{\Phi}_{-q}) \quad \forall\ \omega \in \Omega.$$

Moreover, using (B')

$$E \sup_{0\leq t\leq T} \|M_t(\omega) - f_t^n(\omega)\|_{-q}^2 \leq$$

$$E \sup_{0\leq t\leq T} \sum_{k=n+1}^{\infty} (\bar{M}_t(\phi_k,\omega))^2 <$$

$$\sum_{k=n+1}^{\infty} E \sup_{0\le t\le T} (\bar{M}_t(\phi_k,\ ))^2 \le \sum_{k=n+1}^{\infty} C_T \|\phi_k\|_r^2$$

$$\to 0 \text{ since } \sum_{k=1}^{\infty} \|\phi_k\|_r^2 < \infty.$$

By the Riesz-Fisher theorem there is $U_T \in \mathcal{F}$ with $P(U_T) = 1$ and a subsequence f_{n_k} such that

$$\sup_{0\le t\le T} \|M_t(\ ) - f_t^{n_k}(\ )\|_{-q}^2 \xrightarrow[k\to\infty]{} 0 \quad \forall \quad \in U_T.$$

Since $f^n(\omega) \in D([0,T],\Phi_{-q})\ \forall\ n \in \mathbb{N}\ \forall\ \omega \in \Omega$ this implies that

$$M^T(\omega) \in D([0,T];\Phi_{-q}) \quad \forall\ \omega \in U_T.$$

Now, let $T_n \uparrow \infty$ and put $U = \bigcap_{n\ge 1} U_{T_n}$. Then $P(U) = 1$ and

$$t \to M_t(\omega) \text{ is } \|\cdot\|_{-q}\text{-CADLAG on } [0,\infty) \quad \forall\ \omega \in U.$$

Thus it only remains to show that

$$M_t[\phi] = \bar{M}_t(\phi) \quad \forall\ t\ge 0 \quad P\text{-a.s.} \quad \forall\ \phi \in \Phi:$$

Let $\phi \in \Phi$. Then, for $\ \in G$ we have, for a fixed $t\ge 0$,

$$M_t(\omega)[\phi_k] = \bar{M}_t(\phi_k,\omega) \quad \forall\ k \in \mathbb{N}$$

so by (L) we get

$$M_t[\phi] = \bar{M}_t(\phi) \quad P.a.s. \quad \forall\ \phi \in \text{span}\{\phi_k : k \in \mathbb{N}\}.$$

By (L) and (B) $\phi \to \bar{M}_t(\phi)$ is a continuous linear map from Φ into $L^2(\Omega,\mathcal{F},P)$. Since also M_t is continuous on Φ and since $\text{span}\{\phi_k : k \in \mathbb{N}\}$ is dense in Φ it follows that

$$M_t[\phi] = \bar{M}_t(\phi) \quad P\text{—a.s.}$$

But $t \to M_t[\phi]$ is CADLAG P—a.s. and so is $t \to \bar{M}_t(\phi)$.

Hence

$$M_t[\phi] = \bar{M}_t(\phi) \quad \forall\ t \geq 0 \quad P\text{—a.s.},$$

and since $\phi \in \Phi$ was arbitrary the proof is complete.

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REMARK 7:

Suppose that $\{\bar{M}^n(\phi) : \phi \in \Phi\}_{n\geq 1}$ are families of real valued valued random variables each satisfying (L) and (C) of theorem III.1.12. and each satisfying (B′), but with the same r for every $n \in \mathbb{N}$. Then, since for each n $q = \min\{p : i_p^r \text{ is Hilbert—Schmidt}\}$, we see that q can be chosen independently of n; in other words there is $q \in \mathbb{N}_o$ such that

$$M^{n,T} \in D([0,T],\Phi_{-q}) \quad (P\text{—a.s.}) \quad \forall\ n \in \mathbb{N} \quad \forall\ T \geq 0.$$

We shall now give an example which shows that one cannot always expect to be in the situation discussed by Chari and by Kallianpur & Wolpert, i.e. we shall show that there exist Φ'-valued semimartingales which are not confined to staying in some Φ_{-q} for all t:

EXAMPLE

Let H be a real separable Hilbert space and let L be a positive definite selfadjoint densely defined linear operator on H and suppose that there is some $r_1>0$ such that $(I + L)^{-r_1}$ is Hilbert-Schmidt. Let Φ be the countably Hilbert nuclear space generated by $(I + L)$; i.e.

$$\Phi = \{\phi \in H : \|(I + L)^r\phi\|_H < \infty \quad \forall\, r \in \mathbb{R}\}$$

and for $r \in \mathbb{R}$, $\Phi_r = \|\cdot\|_r$-completion of Φ, where

$$\|\phi\|_r = \|(I + L)^r\phi\|_H; \qquad \phi \in \Phi.$$

Let $p : [0,\infty) \to [0,\infty)$ be an increasing surjective function. Then the mapping $(t,s) \to \langle\phi,\phi\rangle_{p(t\wedge s)}$ is a covariance function for every $\phi \in \Phi$. For each $\phi \in \Phi$ let $\bar{M}_t(\phi)$ be a real Gaussian process with mean zero and covariance

$$E\bar{M}_t(\phi)\bar{M}_s(\phi) = \langle\phi,\phi\rangle_{p(t\wedge s)};$$

(where $\langle\phi,\psi\rangle_r = 1/2(\|\phi + \psi\|_r^2 - \|\phi\|_r^2 - \|\psi\|_r^2) \quad \forall\, r \in \mathbb{R}$).

For each $t\geq 0$ let $r(t)\geq p(t)$ be such that the canonical injection $\iota^{p(t)}_{r(t)} : \Phi_{r(t)} \to \Phi_{p(t)}$ is Hilbert-Schmidt. Let $\{\phi_k^t : k \in \mathbb{N}\}$ be a CONS in $\Phi_{r(t)}$ consisting of elements of Φ and let $\{f_k^t : k \in \mathbb{N}\}$ be the dual CONS in $\Phi_{-r(t)}$. The particular structure of Φ (as generated by $(I + L)$ implies that we may take

$$\phi_j^t = \phi_j/\|\phi_j\|_{r(t)} \text{ and } f_j^t = \frac{\phi_j}{\|\phi_j\|_{-r(t)}}$$

where $\{\phi_j,\lambda_j : j \in \mathbb{N}\}$ is the eigensystem of L and $\|\phi_j\|_r^2 = (1 + \lambda_j)^{2r} \quad \forall\, r \in \mathbb{R}$.

Then, for any $t\geq 0$,

$$E\sum_{k=1}^{\infty}(\bar{M}_t(\phi_k^t))^2 = \sum_{k=1}^{\infty}\|\phi_k^t\|_{p(t)}^2 < \infty$$

In particular, for every $t \geq 0$ there is $\Omega_t \in \mathcal{F}$ with $P(\Omega_t) = 1$ such that

$$\sum_{k=1}^{\infty}(\bar{M}_t(\phi_k^t,\omega))^2 < \infty \quad \forall\, \omega \in \Omega_t.$$

For each $t\geq 0$ define

$$M_t(\omega) = \begin{cases} \sum_{k=1}^{\infty} \bar{M}_t(\phi_k^t,\omega) f_k^t & \text{if } \omega \in \Omega_t \\ 0 & \text{if } \omega \notin \Omega_t \end{cases}$$

Then, for each t, M_t is a $\Phi_{-r(t)}$-valued Gaussian random variable with mean zero. Since $\mathcal{B}(\Phi')$ relativized to Φ_{-r} is equal to $\mathcal{B}(\Phi_{-r})$ for all $r \geq 0$, $(M_t)_{t \geq 0}$ is a Φ'-valued random variable. Moreover, for $\phi \in \Phi$ and $t,s \geq 0$

$$EM_t[\phi]M_s[\phi] =$$

$$E\left(\sum_{j=1}^{\infty} \bar{M}_t(\phi_j^t) < \phi,\phi_j^t >_{r(t)}\right)\cdot\left(\sum_{k=1}^{\infty} \bar{M}_s(\phi_j^s) < \phi,\phi_j^s >_{r(s)}\right) =$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} <\phi,\phi_j^t>_{r(t)} <\phi,\phi_k^s>_{r(s)} E(\bar{M}_t(\phi_j^t)\bar{M}_s(\phi_k^s)) =$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} <\phi,\phi_j^t>_{r(t)} <\phi,\phi_k^s>_{r(s)} <\phi_j^t,\phi_k^s>_{p(t\wedge s)}$$

Now, $r(t) \geq p(t) \geq p(t\wedge s)$ and

$r(s) \geq p(s) \geq p(t\wedge s)$, so

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} <\phi,\phi_j^t>_{r(t)} <\phi,\phi_k^s>_{r(s)} <\phi_j^t,\phi_k^s>_{p(t\wedge s)} =$$

$<\phi,\phi>_{p(t\wedge s)}$, i.e.

$$EM_t[\phi]M_s[\phi] = <\phi,\phi>_{p(t\wedge s)}$$

Hence, for $\phi,\psi \in \Phi$ and $t,s\geq 0$

$$EM_t[\phi]M_s[\psi] =$$
$$1/2(EM_t[\phi + \psi]M_s[\phi + \psi] - EM_t[\phi]M_s[\phi]EM_t[\psi]M_s[\psi])$$

$$= \frac{1}{2}(\|\phi + \psi\|_{p(t\wedge s)} - \|\phi\|_{p(t\wedge s)} - \|\psi\|_{p(t\wedge s)})$$

$$= \langle\phi,\psi\rangle_{p(t\wedge s)}.$$

Therefore,

(28) $\forall\ t_1\geq t_2\geq t_3\geq t_4 \quad \forall\ \phi,\psi \in \Phi$:

$$E(M_{t_1}[\phi] - M_{t_2}[\phi])(M_{t_3}[\psi] - M_{t_4}[\psi]) = 0.$$

Let $\mathbb{F}_t := \{M_s[\phi] : \phi \in \Phi,\ 0\leq s\leq t\}\vee\{P\text{-null sets}\}$.

Since $EM_t[\phi] = 0\ \forall\ t\geq 0 \quad \phi \in \Phi$, (28) implies that $(M_t[\phi])_{t\geq 0}$ is a martingale wrt. $(\mathbb{F}_t)_{t>0}$ for every $\phi \in \Phi$.

Since every real-valued martingale has a CADLAG version, it follows that $M = (M_t)_{t\geq 0}$ is a Φ'-valued weak L^2-semimartingale (in fact martingale).

Recall that

$$\phi_k^t = \phi_k/\|\phi_k\|_{r(t)} \quad \forall\ k \in \mathbb{N} \quad \forall\ t \geq 0$$

and that

$$f_k^t = \phi_k / \|\phi_k\|_{-r(t)} \quad \forall\, k \in \mathbb{N} \quad \forall\, t \geq 0.$$

Moreover, $\{\phi_k : k \in \mathbb{N}\}$ is a complete orthogonal system in Φ_r for every $r \in \mathbb{R}$ with $\|\phi_k\|_r = (1 + \lambda_k)^r$; where $0 \leq \lambda_1 \leq \cdots \leq \Lambda_k \uparrow \infty$ as $k \to \infty$ are the eigenvalues of L.

Fix $q \geq 0$. Then for any $t \geq 0$:

$$\|\bar{M}_t\|_{-q}^2 = \sum_{k=1}^{\infty} (M_t[\phi_k^t])^2 \|f_k^t\|_{-q}^2 \qquad \text{(P-a.s.)}$$

$$= \sum_{k=1}^{\infty} \frac{(M_t[\phi_k^t])^2}{\|\phi_k^t\|_{p(t)}^2} \|\phi_k^t\|_{p(t)}^2 \|f_k^t\|_{-q}^2$$

$$= \sum_{k=1}^{\infty} y_k^2 \,(1 + \lambda_k)^{2(p(t)-q)}$$

where $y_k = \dfrac{M_t[\phi_k^t]}{\|\phi_k^t\|_{p(t)}}$.

Since $M_t[\phi_k^t]$ is zero-mean Gaussian with

$$EM_t[\phi_k^t]M_t[\phi_j^t] = \langle \phi_k^t, \phi_j^t \rangle_{p(t)} = \delta_{kj} \, \|\phi_k^t\|_{p(t)}$$

the y_k s are IID N(0,1). But $\lambda_k \to \infty$ as $k \to \infty$ and $p(t) \to \infty$ as $t \to \infty$ so it is clearly impossible that $M_t \in \Phi_{-q}$ (P-a.s.) $\forall\, t \geq 0$.

Our final objective in this section is to show that nuclear spaces and generators of the type considered in [14] and chapter II satisfy our assumption AS.1.:

III.1.13. PROPOSITION

Let H be a real separable Hilbert space. Let $-L$ be a densely defined closed selfadjoint dissipative linear operator on H whose resolvent has a power which is Hilbert-Schmidt.

Let Φ denote the countably Hilbert nuclear space generated by $(I + L)$; (see Appendix); let denote the nuclear topology of Φ.

Then $-L$ maps Φ into Φ and is -continuous and generates a strongly continuous semigroup $\{T_t : t \geq 0\}$ on H satisfying

(a) $T_t \Phi \subset \Phi$

(b) $T_t|_\Phi$ is τ-continuous on Φ $\forall\ t \geq 0$

(c) $t \to T_t \phi$ is τ-continuous $\forall\ \phi \in \Phi$.

PROOF:

Since $-L$ is a dissipative selfadjoint closed densely defined linear operator on H, $-L$ generates a contraction semigroup $\{T_t : t \geq 0\}$ on H (see e.g. A.V. Balakrishnan [2], corollary4.1).

Moreover, since there is $r_1 > 0$ such that $(I + L)^{-r_1}$ is Hilbert-Schmidt on H, H admits a CONS $\{\phi_j : j \in \mathbb{N}\}$ of eigenvectors of L; $L\phi_j = \lambda_j \phi_j \ \forall\ j$ where $0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots$ and $\lambda_n \xrightarrow[n\to\infty]{} \infty$, and, by definition,

$$\Phi = \{\phi \in H : \sum_{j=1}^{\infty} \langle \phi, \phi_j \rangle_H^2 (1 + \lambda_j)^{2r} < \infty \quad \forall\ r \in \mathbb{R}\}.$$

Let $\phi \in \Phi$, and $\psi_n = \sum_{j=1}^{n} \langle \phi, \phi_j \rangle \phi_j$.

Then $\psi_n \to \phi$ in (Φ, τ) and $-L\psi_n = \sum_{j=1}^{n} -\lambda_j \langle \phi, \phi_j \rangle_H \phi_j$.

Let $r \in \mathbb{R}$. Then, for all $m > n$

$$\| \sum_{j=n+1}^{m} -\lambda_j \langle \phi, \phi_j \rangle_H \phi_j \|_r^2 =$$

$$\sum_{j=n+1}^{m} \lambda_j^2 (1 + \lambda_j)^{2r} \langle \phi, \phi_j \rangle_H^2 \leq$$

$$\sum_{j=n+1}^{m} (1 + \lambda_j)^{2r+2} \langle \phi, \phi_j \rangle_H^2 \xrightarrow[m,n\to\infty]{} 0, \text{ since } \phi \in \Phi.$$

Hence $(-L\phi_n) \underset{n\to\infty}{\longrightarrow} \sum_{j=1}^{\infty} -\lambda_j\langle\phi,\phi_j\rangle\phi_j$

in the topology of Φ. But $-L$ is $\|\cdot\|_H$-closed. Since $\|\cdot\|_H$ is continuous on $(\Phi,\ )$ we get

$-L\phi = \sum_{j=1}^{\infty} -\lambda_j\langle\phi,\phi_j\rangle_H\phi_j$ and hence for any $r \in \mathbb{R}$

$$\|-L\phi\|_r^2 = \sum_{j=1}^{\infty} \lambda_j^2\langle\phi,\phi_j\rangle_H^2(1+\lambda_j)^{2r} \quad \forall\ \phi \in \Phi$$

$$\leq \sum_{j=1}^{\infty} (1+\lambda_j)^{2r+2}\langle\phi,\phi_j\rangle_H^2$$

$$\leq \|\phi\|_{r+1}^2.$$

Hence $-L\Phi \subset \Phi$ and $-L$ is τ-continuous on Φ.

Next, with $\phi \in \Phi$, $\phi = \sum_{j=1}^{\infty} \langle\phi,\phi_j\rangle_H\phi_j$ (converging in $\Phi_r\ \forall\ r$)

we have for any $t>0$:

$T_t\phi = \sum_{j=1}^{\infty} e^{-\lambda_j t}\langle\phi,\phi_j\rangle_H\phi_j$, and for $r \in \mathbb{R}$ we have

$$\|T_t\phi\|_r^2 = \sum_{j=1}^{\infty} e^{-2\lambda_j t}\langle\phi,\phi_j\rangle_H^2(1+\lambda_j)^{2r}$$

$$\leq \sum_{j=1}^{\infty} \langle\phi,\phi_j\rangle_H^2(1 + \lambda_j)^{2r} = \|\phi\|_r^2,$$ proving (a) and

(b).

Finally, let $\phi \in \Phi$ and $s \geq 0$. Then, for any $r \in \mathbb{R}$,

$$\|T_t\phi - T_s\phi\|_r^2 = \sum_{j=1}^{\infty} (e^{-\lambda_j t} - e^{-\lambda_j s})^2 \langle\phi,\phi_j\rangle_H^2(1 + \lambda_j)^{2r}$$

$$\leq 4 \sum_{j=1}^{\infty} \langle\phi,\phi_j\rangle_H^2(1 + \lambda_j)^{2r}$$

$$= 4\|\phi\|_r^2 < \infty$$

and since $t \to e^{-\lambda_j t}$ is continuous $\forall\ j \in \mathbb{N}$ the DCT gives

$$\lim_{t \to s} \|T_t\phi - T_s\phi\|_r^2 = 0 \quad \forall\ r \in \mathbb{R}.$$

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III.2. WEAK CONVERGENCE

Several recent articles, including [14] and [4], have investigated special cases of weak convergence of solutions to Φ'-valued linear SDE's. A common characteristic of these articles has been that the authors were concerned with situations in which the limiting process was driven by a Φ'-valued Wiener process (see cqapter II for definition) and in which A is a closed and dissipative operator on H. In addition each author has operated with either a special sequence of noises ([14]; Poisson generated noise converging to Wiener noise) or a particular nuclear space ([4]; $\Phi = \mathscr{S}(\mathbb{R}^d)$). In either case their methods were designed specifically for the problem in question and do not leave ample room for extensions. Here we shall exploit the fact that our method of solution in section III.1. has provided an explicit formula for the solution to a linear Φ'-valued SDE to derive a general weak convergence result, which requires neither a special sequence of noises nor a special structure of Φ and does not restrict attention to the case where the limiting process is driven by Wiener noise. Moreover, we shall not assume that A is dissipative.

The assumptions appearing in our result (theorem III.2.1.) may at first appear rather abstract and perhaps difficult to apply. However, as we shall see in chapter IV, these assumptions together with a result of I. Mitoma [22] translate very easily into explicit conditions when applied to concrete examples. One of the recent applications of this subject has been in neurophysiology, and we shall see in chapter IV how various results in this field as well as new results may be derived with the help of theorem III.2.1..

Let $M = (M_t)_{t \geq 0}$ and $M^n = (M_t^n)_{t \geq 0}$; $n \in \mathbb{N}$ be Φ'-valued weak L^2-semimartingales such that $M^T := (M_t)_{t \in [0,T]}$ and

$M^{n,T} := (M_t^n)_{t\in[0,T]}$ satisfy

<u>AS.3.</u> $\forall\, T>0 \;\exists\; \Omega_T \in \mathcal{F}$ with $P(\Omega_T) = 1$, $\exists\; q_T \in \mathbb{N}_o$:

$\forall\, n \in \mathbb{N} : M^{n,T}(\omega) \in D([0,T],\Phi_{-q_T})$ and

$M^T(\omega) \in D([0,T],\Phi_{-q_T}) \quad \forall\, \omega \in \Omega_T$

<u>and</u>

(29) $\sup\limits_{n\in\mathbb{N}} E \sup\limits_{0\le t\le T} \|M_t^{n,T}\|_{-q_T}^2 < \infty.$

<u>REMARK 8</u>

By proposition III.1.6. and remark 4 (page 101) for each $T>0$ and $n \in \mathbb{N}$ there is $q_T^n \in \mathbb{N}_o$ such that

$M^{n,T} \in D([0,T],\Phi_{-q_T^n}) \quad$ P-a.s. $\quad \forall\, n \in \mathbb{N}$

AS.3. therefore only serves the purpose of securing that the same q_T will do for all $n \in \mathbb{N}$.

Let η^n, η be Φ'-valued random variables satisfying

(30) $\exists\, r_1 \in \mathbb{N} : \sup\limits_{n\in\mathbb{N}} \max\{E\|\eta^n\|_{-r_1}^2, E\|\eta\|_{-r_1}^2\} < \infty.$

Let $\xi^n = (\xi_t^n)_{t\ge 0}$ respectively $\xi = (\xi_t)_{t\ge 0}$ denote the unique solution satisfying (10) to

$$d\xi_t^n = A' \,{}_t^n dt + dM_t^n; \ \xi_o^n = \eta^n$$

respectively,

$$d\xi_t = A'\xi_t dt + dM_t; \ \xi_o = \eta$$

whose existence is guaranteed by theorem III.1.5.. For each T>0 let, as usual,

$$\xi^{n,T} := (\xi_t^n)_{t\in[0,T]}$$

$$\xi^T \quad := (\xi_t)_{t \in [0,T]}.$$

By AS.3. there is, for every T>0, $q_T \in \mathbb{N}_o$ such that, with probability one,

$$M^T, M^{n,T} \in D([0,T], \Phi_{-q_T}), \quad \forall\ n \in \mathbb{N}$$

and by an argument very similar to that employed in the proof of theorem III.1.4. it may be seen that this implies that for each T>0 there is a $p_T \in \mathbb{N}_o$ such that

$$\xi^{n,T} \in D([0,T], \Phi_{-p_T}) \quad \text{P-a.s.} \quad \forall\ n \in \mathbb{N} \text{ and}$$

$$\xi^T \in D([0,T], \Phi_{-p_T}) \quad \text{P-a.s.}$$

Recalling that the operator A is required to satisfy AS.1. (page 58) we can now state the main result:

<u>THEOREM III.2.1.</u>

Let $M = (M_t)_{t\geq 0}$ and $M^n = (M^n_t)_{t\geq 0}$ satisfy AS.3. and suppose that η^n and η satisfy (30).

Let T>0 and suppose that

$M^{n,T} \Rightarrow M^T$ on $D([0,T],\Phi_{-q_T})$ as $n \to \infty$

and that

$P\cdot(\eta^n)^{-1} \Rightarrow P\circ\eta^{-1}$ as $n \to \infty$. Then

$\xi^{n,T} \Rightarrow \xi^T$ on $D([0,T],\Phi_{-p_T})$ as $n \to \infty$.

—Before proceeding to the proof we need some lemmae:

<u>III.2.2. LEMMA</u>

Let T>0 and let V_T denote the set of all real valued functions defined on [0,T]. Define a mapping G:

$\Phi \times D([0,T],\Phi_{-q_T}) \to V_T$ (q_T is given from AS.3.) by

$G(\phi,F) := v, \quad \phi \in \Phi, \quad F \in D([0,T],\Phi_{-q_T})$ where

$v(t) = \int_0^t F_s[T_{t-s}A\phi]ds; \quad t \in [0,T]$. Then

(A) $\forall\ \phi \in \Phi\ \forall\ F \in D([0,T],\Phi_{-q_T}) : G(\phi,F) \in C([0,T],\mathbb{R})$

(where $C([0,T],\mathbb{R})$ is the space of all continuous functions $f : [0,T] \to \mathbb{R}$ equipped with the usual topology)

(B) $\forall\ \phi \in \Phi : G(\phi,.) : D([0,T],\Phi_{-q_T}) \to C([0,T],\mathbb{R})$ is continuous

(C) $\forall\ F \in D([0,T],\Phi_{-q_T})\ \forall\ t \in [0,T] : G(.,F)(t) \in \Phi'$.

<u>PROOF:</u>

Let $\phi \in \Phi$ and $F \in D([0,T],\Phi_{-q_T})$. By AS.1. and CADLAG-property of F wrt. $\|\cdot\|_{-q_T}$ it is easily seen that $s \to F_s[T_{t-s}A\phi]$ is CADLAG on $[0,t]$ for any $t \in (0,T]$. In particular, this mapping is integrable over $[0,t]$ for any $t \in (0,T]$ and hence G is well-defined.

(A) Let $\phi \in \Phi$ and $F \in D([0,T],\Phi_{-q_T})$. Fix $u \in [0,T]$. We have to show that $t \to \int_0^t F_s[T_{t-s}A\phi]$ is continuous at u:

$$|\int_0^t F_s[T_{t-s}A\phi] - \int_0^u F_s[T_{u-s}A\phi]ds| =$$

$$|\int_0^{t\wedge u} (F_s[T_{t-s}A\phi] - F_s[T_{u-s}A\phi])ds +$$

$$\operatorname{sgn}(t-u)\int_{t\wedge u}^{t\vee u} F_s[T_{t\vee u-s}A\phi]ds| \le$$

$$|\int_0^{t\wedge u} (F_s[T_{t-s}A\phi] - F_s[T_{u-s}A\phi]ds| +$$

$$|\int_{t\wedge u}^{t\vee u} F_s[T_{t\vee u-s}A\phi]ds| \le$$

$$\int_0^t (\sup_{s\in[0,T]} \|F_s\|_{-q_T})\|T_{t-s}A\phi - T_{u-s}A\phi\|_{q_T}ds +$$

$$\int_{t\wedge u}^{t\vee u} (\sup_{s\in[0,T]} \|F_s\|_{-q_T})\|T_{t\vee u-s}A\phi\|_{q_T}ds$$

(letting $L := \sup_{s\in[0,T]} \|F_s\|_{-q_T}$, we have $L < \infty$ by CADLAG-

property of F wrt. $\|.\|_{-q_T}$ and thus)

$$\le \int_0^T L\, 1_{[0,t\ u]}(s)\|T_{t-s}A\phi - T_{u-s}A\phi\|_{q_T}ds +$$

$$\int_{t\wedge u}^{t\vee u} L\ \|T_{t\vee u-s}A\phi\|ds$$

-The first term tends to zero as $t \to u$ by the DCT, since for $s \in [0,T]$

$$1_{[0,t\ u]}(s)\|T_{t-s}A\phi - T_{t-u}A\phi\|_{q_T} \xrightarrow[t\to u]{} 0 \text{ by AS>1. (c) and}$$

$$1_{[0,t\ s]}(s)\|T_{t-s}A\phi - T_{u-s}A\phi\|_{q_T} \le$$

$$2\sup_{0\le s\le t\le T} \|T_{t-s}A\phi\|_{q_t} < \infty, \quad \text{since}$$

$(s,t) \to \|T_{t-s}A\phi\|_{q_T}$ is continuous on

$\{(s,t) : 0 \le s \le t \le T\}$ as a consequence of AS.1. (c).

—The second term also tends to zero as $t \to u$, since

$$\|T_{t\ u-s}A\phi\|_{q_T} \cdot 1_{[t\ u,t\ u]}(s) \le$$

$$\sup_{0\le s\le t\le T} \|T_{t-s}A\phi\|_{q_T} < \infty \text{ and since}$$

$$t\vee u - t\wedge u = |t-u|.$$

This concludes the proof of (A).

(B) Let $\phi \in \Phi$, and let $F^n \to F$ in $D([0,T],\Phi_{-q_T})$. Then

$$\sup_{t\in[0,T]} |G(\phi,F)(t) - G(\phi,F^n)(t)| =$$

$$\sup_{t\in[0,T]} \left|\int_0^t (F_s-F_s^n)[T_{t-s}A\phi]ds\right| \le$$

$$\sup_{t\in[0,T]} \int_0^t \|F_s - F_s^n\|_{-q_T} \|T_{t-s}A\phi\|_{q_T} ds$$

Now $F^n \to F$ in $D([0,T],\Phi_{-q_T})$ implies that

$$(31) \qquad K = \sup_{n\in\mathbb{N}} \sup_{0\le s\le T} \|F_s - F_s^n\|_{-q_T} < \infty$$

(see e.g. (14.32) in theorem 14.2 of Billingsley [3]. His proof for the case $T = 1$ and $\Phi_{-q_T} = \mathbb{R}$ extends without change to $T>0$ and any real separable Hilbert space).

Moreover, convergence of F^n to F in $D([0,T],\Phi_{-q_T})$ implies that $\|F_s^n - F_s\|_{-q_T}$ tends to zero at any continuity point of F. Since the set of discontinuities of $F \in D([0,T],\Phi_{-q_T})$ has Lebesque measure zero and since

$$\sup_{t\in[0,T]} \int_0^t \|F_s - F_s^n\|_{-q_T} \|T_{t-s}A\phi\|_{q_T} ds \leq$$

$$(\sup_{0\leq s\leq t\leq T} \|T_{t-s}A\phi\|_{q_T}) \int_0^t \|F_s - F_s^n\|_{-q_T} ds,$$

(31) and the DCT gives

$$\sup_{t\in[0,T]} |G(\phi,F)(t) - G(\phi,F^n)(t)| \xrightarrow[n\to\infty]{} 0 \text{ proving (B).}$$

(C) Let $F \in D([0,T],\Phi_{-q_T})$ and $t \in [0,T]$. Then

$$\Phi \ni \phi \to G(\phi,F)(t) = \int_0^t F_s[T_{t-s}A\phi]ds$$

is obviously linear. Let $\phi_n \to \phi$ in $(\Phi,\mathcal{T})$ then for each $s \in [0,T]$

$F_s[T_{t-s}A\phi_n] \to F_s[T_{t-s}A\phi]$, by AS.1. (b), continuity of A on Φ and the fact that $F_s \in \Phi' \ \forall\ s \in [0,T]$.

Also, $\phi \to \|T_{t-s}A\phi\|_{q_T}$ is continuous on Φ and therefore

$$f(s) := \sup_{n\in\mathbb{N}} \|T_{T-s}A(\phi_n - \phi)\|_{q_T} < \infty \quad \forall\ s \in [0,T],$$

and since $s \to \|T_{t-s}A(\phi_n-\phi)\|_{q_T}$ is continuous for each

$n \in \mathbb{N}$, f is a lower-semicontinuous function of $s \in [0,T]$. In particular, f is bounded on $[0,T]$. Hence

$$|F_s[T_{t-s}A\phi_n] - F_s[T_{t-s}A\phi]| \leq$$

$$\|F_s\|_{-q_T}\|T_{t-s}A(\phi_n - \phi)\|_{q_T} \leq \|F_s\|_{-q_T}f(s)$$

$\in L^{\infty}([0,T]) \subset L^1([0,t])$ (recall that

$s \to \|F_s\|_{-q_T}$ is CADLAG and hence bounded on compact intervals). Therefore, the DCT yields

$$|G(\phi_n,F)(t) - G(\phi,F)(t)| \leq$$

$$\int_0^t |F_s[T_{t-s}A\phi_n] - F_s[T_{t-s}A\phi]|ds \xrightarrow[n\to\infty]{} 0,$$

concluding the proof.

$\|$

III.2.3.LEMMA

Let $M^{n,T}$, M^T be as in theorem III.2.1.. Let $T>0$, and $K : D([0,T],\Phi_{-q_T}) \to C([0,T],\mathbb{R})$ be continuous. Then

$$P\,(K(M^{n,T}))^{-1} \underset{n\to\infty}{\Longrightarrow} P\,(K(M^T))^{-1}.$$

PROOF:

Both $D([0,T],\Phi_{-q_T})$ and $C([0,T],\mathbb{R})$ are complete metric spaces, $P\circ(M^{n,T})^{-1} \underset{N\to\infty}{\Longrightarrow} P\circ(M^T)^{-1}$ by assumption, and

$K : D([0,T],\Phi_{-q_T}) \to C([0,T],\mathbb{R})$ is continuous.

Hence the conclusion follows from (e.g.) Billingsley [3], theorem 5.1. page 30.

DEFINITION

Following I. Mitoma [22] (page 997) we say that a sequence $\{P_n\}$ of probability measures on $D([0,T],\Phi')$ is uniformly k-continuous if

$$\forall \epsilon > 0 \; \forall \rho > 0 \; \exists \delta > 0 : P_n\{X \in D([0,T],\Phi') : \sup_{t\in[0,T]} |X_t[\phi]| > \epsilon\} \leq \rho \quad \forall n \geq 1 \text{ whenever } \|\phi\|_k \leq \delta.$$

Similarly, we say that a sequence $(X^n)_{n\geq 1}$ of $D([0,T],\Phi')$-valued random variables is uniformly k-continuous if $P_n := P\,(X^n)^{-1}$; $n \geq 1$ is uniformly k-continuous.

($D([0,T],\Phi')$ is defined by Mitoma [22] and contains $D([0,T],\Phi_{-q})$ $\forall q \geq 0$).

Mitoma [22] (theorem 4.1 and remark (R.1.)) has proved the following result which we restate for the convenience of the reader:

THEOREM A (MITOMA)

Suppose that the sample paths of Y^n, $n\geq 1$ are in $D([0,T],\Phi_{-p})$ and that $(Y^n)_{n\geq 1}$ is uniformly k-continuous for some $k\geq p$. Suppose further that for every $\phi \in \Phi$ the sequence of distributions of $Y^n[\phi]$ is tight in $D([0,T],\mathbb{R})$.

Then $\{Y^n : n \geq 1\}$ is tight on $D([0,T],\Phi_{-p})$.

III.2.4. LEMMA:

Let $T > 0$, $p \geq 0$. Let $\mathcal{K}$ denote the class of sets

$$\{\{x \in D([0,T],\Phi_{-p} : x[\phi] \in A\} : \phi \in \Phi,\ A \in \mathcal{B}(D[0,T],\mathbb{R}))\}.$$

Then $\sigma(\mathcal{K}) = \mathcal{B}(D[0,T],\Phi_{-p}))$.

PROOF:

Recall that the metric on $D([0,T],\Phi_{-p})$ is (see e.g. appendix in [14]) given by

$$d(x,y) = \inf_{\lambda,\mu\in\Lambda_T} \max\{\sup_{0\leq t\leq T} \|x_{\lambda(t)} - Y_{\mu(t)}\|_{-p},\ \delta_T(\mu,\lambda)\}$$

where Λ_T denotes the set of all strictly increasing surjective functions $[0,T] \to [0,T]$, and where the metric δ_T on Λ_T is defined by

$$\delta_T(\lambda,\mu) = \sup_{0 \le s < t \le T} \left|\log \frac{\lambda(t) - \lambda(s)}{\mu(t) - \mu(s)}\right|.$$

Similarly, (see e.g. Billingsley, [3]) the metric on $D([0,T],\mathbb{R})$ is

$$d_{\mathbb{R}}(f,g) = \inf_{\lambda,\mu \in \Lambda_T} \max\{\sup_{0 \le t \le T} |f\ \lambda(t) - g\ \mu(t)|, \delta_T(\mu,\lambda)\}$$

–It is sufficient to show that

$$\forall\ y \in D([0,T],\Phi_{-p}) \quad \forall\ \epsilon > 0:$$

$$\{x \in D([0,T],\Phi_{-p}) : d(x,y) < \epsilon\} \in \sigma(\mathcal{H}).$$

To do this, we first show that for any $x,y \in D([0,T]\Phi_{-p})$ we have:

$$(32) \qquad \inf_{\phi \in B^c} d_{\mathbb{R}}(x[\phi],y[\phi]) = d(x,y)$$

where $B = \{\phi \in \tilde{\Phi} : \|\phi\|_p \le 1\}$ and $\tilde{\Phi}$ is a countable dense set in Φ_p such that $\tilde{\Phi}$ Φ (recall that Φ_p is separable), and where $x[\phi]$ denotes the function $h \in D([0,T),\mathbb{R})$ given by

$$h(t) = x_t[\phi];\ t \in [0,T].$$

Let $x,y \in D([0,T],\Phi_{-p})$.

Then

$$\inf_{\phi \in B^c} d_{\mathbb{R}}(x[\phi],y[\phi]) =$$

$$\inf_{\phi \in B^c} \inf_{\lambda,\mu \in \Lambda_T} \max\{\sup_{0 \le t \le T} |x_{\lambda(t)}[\phi] - y_{\mu(t)}[\phi]|, \delta_T(\mu,\lambda)\} =$$

$$\inf_{\lambda,\mu \in \Lambda_T} \inf_{\phi \in B^c} \max\{\sup_{0 \le t \le T} |x_{\lambda(t)}[\phi] - y_{\mu(t)}[\phi]|, \delta_T(\mu,\lambda)\}$$

Define $f_{\mu,\lambda}(\phi) = \sup_{0 \le t \le T} |x_{\lambda(t)}[\phi] - y_{\mu(t)}[\phi]|$.

Baire's theorem implies that f is $\|\cdot\|_p$-continuous.

Moreover,

$f_{\mu,\lambda}(a\phi) = |a| f_{\mu,\lambda}(\phi) \quad \forall\ a \in \mathbb{R}$. Hence

(33) $\qquad \inf_{\phi \in B^c} f_{\mu,\lambda}(\phi) = \sup_{\phi \in B} f_{\mu,\lambda}(\phi), \qquad$ so

$$\inf_{\phi \in B^c} d_{\mathbb{R}}(x[\phi],y[\phi]) =$$

$$\inf_{\lambda,\mu \in \Lambda_T} \inf_{\phi \in B^c} \max\{f_{\mu,\lambda}(\phi), \delta_T(\phi)\} =$$

$$\inf_{\lambda,\in \Lambda_T} \max\{\inf_{\phi \in B^c} f_{\mu,\lambda}(\phi), \delta_T(\phi)\} = \quad \text{(by (33))}$$

$$\inf_{\lambda,\mu \in \Lambda_T} \max\{\sup_{\phi \in B} f_{\mu,\lambda}(\phi), \delta_T(\phi)\} =$$

$$\inf_{\lambda,\mu\in\Lambda_T} \max\{\sup_{\phi\in B} \sup_{0\le t\le T} |x_{\lambda(t)}[\phi] - y_{\mu(t)}[\phi]|, \delta_T(\mu,\lambda)\} =$$

$$\inf_{\lambda,\mu\in\Lambda_T} \max\{\sup_{0\le t\le T} \sup_{\phi\in B} |x_{\lambda(t)}[\phi] - y_{\mu(t)}[\phi], \delta_T(\mu,\lambda)\} =$$

$$\inf_{\lambda,\mu\in\Lambda_T} \max\{\sup_{0\le t\le T} \|x_{\lambda(t)} - y_{\mu(t)}\|_{-p}, \delta_T(\mu,\lambda)\} =$$

$d(x,y)$, and (32) is proved.

Hence, for fixed $y \in D([0,T],\Phi_{-p})$ and $\epsilon > 0$

$$\{x \in D([0,T],\Phi_{-p}) : d(x,y) < \epsilon\} =$$

$$\{x \in D([0,T],\Phi_{-p}) : \inf_{\phi\in B^c} d_{\mathbb{R}}(x[\phi],y[\phi]) < \epsilon\} =$$

$$\{x \in D([0,T],\Phi_{-p}) : \inf_{\phi\in B^c} d_{\mathbb{R}}(x[\phi],y[\phi]) \ge \epsilon\}^c =$$

$$\left[\bigcap_{\phi\in B^c} \{x \in D([0,T],\Phi_{-p}) : d_{\mathbb{R}}(x[\phi],y[\phi]) \ge \epsilon\}\right]^c \in \quad (\quad).$$

//

PROPOSITION III.2.5.

Let $T > 0$ and $q \ge 0$. Let $\{X^n : N \in \mathbb{N}\}$ be a tight sequence of $D([0,T],\Phi_{-p})$-valued random variables.
Let X be a $D([0,T],\Phi_{-p})$-valued random variable. Then

$$X^n \underset{n\to\infty}{\Longrightarrow} X \text{ on } D([0,T],\Phi_{-p}) \quad \text{iff}$$

$$X^n[\phi] \underset{n\to\infty}{\Longrightarrow} X[\phi] \text{ on } D([0,T],\mathbb{R}) \quad \forall\ \phi \in \Phi.$$

PROOF:

Necessity follows from Billingsley [3], theorem 5.1. since for each $\phi \in \Phi$, the map $H_\phi : D([0,T],\Phi_{-p}) \to D([0,T],\mathbb{R})$ given by $H_\phi(x) = x[\phi]$ is continuous, and since both $D([0,T],\Phi_{-p})$ and $D([0,T],\mathbb{R})$ are complete metric spaces.

Sufficiency: Since $D([0,T],\Phi_{-p})$ is a complete metric space, tightness of $\{X^n : n \in \mathbb{N}\}$ implies relative compactness by Prohorov's theorem.

Let $P\,Y^{-1}$ be any limit point of $\{P\,(X^n)^{-1} : n \in \mathbb{N}\}$. Then there is a subsequence $\{X^{n_k} : k \in \mathbb{N}\}$ such that $X^{n_k} \underset{k\to\infty}{\Longrightarrow} Y$.

Since, for each $\phi \in \Phi$, H_ϕ is continuous, this implies that

$$X^{n_k}[\phi] \underset{k\to\infty}{\Longrightarrow} Y[\phi] \quad \forall\, \phi \in \Phi.$$

But by assumption $X^{n_k}[\phi] \underset{k\to\infty}{\Longrightarrow} X[\phi] \quad \forall\, \phi \in \Phi.$

Hence $P\circ(Y[\phi])^{-1} = P\circ(X[\phi])^{-1} \quad \forall\, \phi \in \Phi$

i.e. $P(Y[\phi] \in A) = P(X[\phi] \in A) \quad \forall\, \phi \in \Phi$
$\forall\, A \in \mathcal{B}(D[0,T],\mathbb{R})$.

By lemma III.2.4. this implies that

$P\circ Y^{-1} = P\circ X^{-1}$, and so $P\circ X^{-1}$ is the unique limit point of

$\{P\circ(X^n)^{-1} : n \in \mathbb{N}\}$. But then, since $\{P\circ(X^n)^{-1} : n \in \mathbb{N}\}$ is relatively compact, we must have

$$P\circ(X^n)^{-1} \underset{n\to\infty}{\Longrightarrow} P\cdot X^{-1}.$$

(For if not, then there is a subsequence $\{n_k : k \in \mathbb{N}\}$ and a probability measure $R \neq P\circ X^{-1}$ on $D([0,T],\Phi_{-p})$ such that $P\circ(X^{n_k})^{-1} \underset{k\to\infty}{\Longrightarrow} R$, contradicting uniqueness of $P\,X^{-1}$ as a limit point).

$\natural$

III.2.6. COROLLARY:

Let $T > 0$ and $p \geq 0$. Let X^n, X be $D([0,T],\Phi_{-p})$-valued random variables such that $\{X^n : n \geq 1\}$ is uniformly k-continuous for some $k \geq p$. Then

(a) $\quad X^n \underset{n\to\infty}{\Longrightarrow} X$ on $D([0,T],\Phi_{-p})$

iff

(b) $\quad X^n[\phi] \underset{n\to\infty}{\Longrightarrow} X[\phi]$ on $D([0,T],\mathbb{R}) \quad \forall\, \phi \in \Phi$.

PROOF:

(a)=>(b): Since $D([0,T],\Phi_{-p})$ and $D([0,T],\mathbb{R})$ are complete metric spaces and the map

$$D([0,T],\Phi_{-p}) \ni x \to x[\phi] \in D([0,T],\mathbb{R})$$

is continuous for every $\phi \in \Phi_p$, (a) implies (b) by Billingsley [3] theorem 5.1.

<u>(b)=>(a):</u> Since $X^n[\phi] \underset{n\to\infty}{\Longrightarrow} X[\phi]$ for every $\phi \in \Phi$, $\{X^n[\phi] : n \geq 1\}$ is tight for each $\phi \in \Phi$ and thus by uniform k-continuity and the quoted theorem of Mitoma $\{X^n : n \geq 1\}$ is tight. Hence the assumptions of proposition III.2.5. are satisfied and the conclusion now follows from proposition III.2.5..

$\|$

Now we can prove Theorem III.2.1.:

<u>PROOF OF THEOREM III.2.1.:</u>

By corollary III.2.6. we must show that

(i) For every $\phi \in \Phi$ the sequence $\xi^{n,T}[\phi]$ converges weakly on $D([0,T],\mathbb{R})$ to $\xi^T[\phi]$

(ii) $\exists k \geq p_T : \forall \epsilon > 0 \;\forall \rho > 0 \;\exists \delta > 0$

$$P(\sup_{0\leq t\leq T} |\xi_t^{n,T}[\phi]| > \epsilon) \leq \rho \text{ whenever } \|\phi\|_k \leq \delta.$$

<u>(i):</u> Let $\phi \in \Phi$. Then, letting $y.$ denote the function $t\to y_t$,

$$\cdot^{n,T}[\phi] = \eta^n[T.\phi] + M_\cdot^{n,T}[\phi] + G(\phi, M^{n,T})(.) \text{ and}$$

$$\cdot T[\phi] = \eta[T.\phi] + M_\cdot^T[\phi] + G(\phi, M^T)(.),$$

where G is as in Lemma III.2.2. Let Q_1^n (respectively Q_1) denote the measure induced on $C([0,T],\mathbb{R}) \subset D([0,T],\mathbb{R})$ by $\eta^n[T.\phi]$ (respectively by $\eta[T.\phi]$) and let Q_2^n (respectively Q_2) denote the measure induced on $D([0,T],\mathbb{R})$ by $M_\cdot^{n,T}[\phi]$ (respectively by $M_\cdot^T[\phi]$) and let Q_3^n (respectively Q_3) denote the measure induced on $C([0,T],\mathbb{R})$ by $G(\phi,M^{n,T})$ (respectively by $G(\phi,M^T)$) (recall (A) of LEMMA III.2.2.).

By Kallianpur & Wolpert [14], Corollary 3.1. (page 142) it is sufficient to prove that

(iv) $\quad Q_i^n \Rightarrow Q_i$ as $n \to \infty$; $i = 1,2,3$.

<u>i = 3:</u> By lemma III.2.2. (A) and (B) $G(\phi,.) : D([0,T],\Phi_{-q_T}) \to C([0,T],\mathbb{R})$ is continuous. By AS.3. $M^{n,T}$, $M^T \in D([0,T],\Phi_{-q_T})$ (P–a.s.). Since $M^{n,T} \underset{n\to\infty}{\Longrightarrow} M^T$ on $D([0,T],\Phi_{-q_T})$ by assumption $Q_3^n \underset{N\to\infty}{\Longrightarrow} Q_3$ by Lemma III.2.3..

<u>i = 2:</u> Is an immediate consequence of Billingsley [3], theorem 5.1. (page 30) and the assumption that $M^{n,T} \underset{n\to\infty}{\Longrightarrow} M^T$ (the mapping $K : D([0,T],\Phi_{-q_T}) \to D([0,T],\mathbb{R})$ given by

$$K(F)(t) = F_t[\phi];\ t \in [0,T]$$

is continuous and both $D([0,T],\Phi_{-q_T})$ and $D([0,T],\mathbb{R})$ are complete metric spaces).

i = 1: This follows from the assumption that $\eta^n \underset{n\to\infty}{\Longrightarrow} \eta$ and Billingsley [3], theorem 5.1. (page 30), since for each $\phi \in \Phi$ the mapping $H : \Phi_{-r_1} \to C([0,T],\mathbb{R})$ defined by

$$H(\eta) = h \text{ where}$$

$$h(t) = \eta[T_t\phi];\ \eta \in \Phi_{-r_1}$$

is continuous and both Φ_{-r_1} and $C([0,T],\mathbb{R})$ are complete metric spaces.

This concludes the proof of (i).

(ii): Since

$$\xi_t^{n,T}[\phi] = \eta^n[T_t\phi] + \int_0^t M_s^{n,T}[T_{t-s}A\phi]ds + M_t^{n,T}[\phi]$$

we get (using Schwartz inequality)

$$|\xi_t^{n,T}[\phi]|^2 \le 3|\eta^n[T_t\phi]|^2 + 3t \int_0^t |M_s^{n,T}[T_{t-s}A\phi]|^2 ds$$

$$+ 3|M_t^{n,T}[\phi]|^2$$

$$\leq 3|\eta^n[T_t\phi]|^2 + 3t^2 \sup_{0\leq s\leq t} |M_s^{n,T}[T_{t-s}A\phi]|^2 + 3|M_t^{n,T}[\phi]|^2$$

Thus

$$E \sup_{0\leq t\leq T} |\xi_t^{n,T}[\phi]| \leq 3(E \sup_{0\leq t\leq T} |\eta^n[T_t\phi]|^2 +$$

$$T^2 E \sup_{0\leq t\leq T} \sup_{0\leq s\leq t} |M_s^{n,T}[T_{t-s}A\phi]|^2 + E(\sup_{0\leq t\leq T} |M_t^{n,T}[\phi]|^2))$$

$$\leq 3((\sup_{0\leq t\leq T} \|T_t\phi\|_{r_1}^2)E\|\eta^n\|_{-r_1}^2 + T^2(\sup_{0\leq s\leq t\leq T} \|T_{t-s}A\phi\|_{q_T}^2)$$

$$E \sup_{0\leq t\leq T} \|M_t^{n,T}\|_{-q_T}^2 + E \sup_{0\leq t\leq T} \|M_t^{n,T}\|_{-q_T}^2 \|\phi\|_{q_T}^2)$$

By assumption (30) $E\|\eta^n\|_{-r_1}^2 \leq C_1 \ \forall \ n\geq 1$ for some $C_1 \in [0,\infty)$ and by (29) of AS.3.

$$E \sup_{0\leq t\leq T} \|M_t^{n,T}\|_{-q_T}^2 \leq K < \infty \quad \forall \ n\geq 1$$

for some $K>0$. Hence

$$E(\sup_{t\in[0,T]} |\xi_t^{n,T}[\phi]|^2) \leq 3(C_1 \sup_{0\leq t\leq T} \|T_t\phi\|_{r_1}^2 +$$

$$KT^2 \sup_{0\leq s\leq t\leq T} \|T_{t-s}A\phi\|_{q_T}^2 + K\|\phi\|_{q_T}^2)$$

$$\leq 3(C_1 \sup_{0\leq t\leq T} \|T_t\phi\|_{r_1}^2 + KT^2 \sup_{0\leq t\leq T} \|T_tA\phi\|_{q_T}^2 + K\|\phi\|_{q_T}^2)$$

Let $g_1(\phi) := \sup_{0\leq t\leq T} \|T_t\phi\|_{r_1}^2$ and

$$g_2(\phi) := \sup_{0\leq t\leq T} \|T_tA\phi\|_{q_T}^2; \qquad \phi \in \Phi$$

Since $t \to \|T_t A\phi\|_r$ is continuous on Φ for any $r \geq 0$, g_i, $i = 1,2$ is a lower semicontinuous convex function of $\phi \in \Phi$ satisfying $g_i(a\phi) = |a|^2 g_i(\phi) \quad \forall\, a \in \mathbb{R}$. Hence Baire's theorem (c.f. proof of theorem III.1.4. page) yields the existence of constants C_2 and C_3 and $r_2, r_3 \in \mathbb{N}_o$ such that

$$g_i(\phi) \leq C_{i+1}\|\phi\|^2_{r_{i+1}}; \quad i = 1,2.$$

Let $k = r_2 \vee r_3 \vee q_T \vee p_T$. Then

$$E \sup_{0 \leq t \leq T} |\,{}^{n,T}_{t}[\phi]|^2 \leq 3(C_1.C_2 + T^2KC_3 + K)\|\phi\|^2_k \quad \forall\, n \geq 1,$$

and thus by Chebyshev's inequality

$$P(\sup_{0 \leq t \leq T} |\xi^{n,T}_t[\phi]| > \epsilon) \leq$$

$$\epsilon^{-2}3(C_1C_2 + T^2KC_3 + K)\|\phi\|^2_k \quad \forall\, n \geq 1$$

and therefore choosing $0 < \delta^2 < \epsilon^2(3(C_1C_2 + T^2KC_3 + K)^{-1}\rho$ we see that $\xi^{n,T}$ is uniformly k-continuous. Since $k \geq p_T$, this completes the proof.

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REMARK 9:

Mitoma's result (theorem A) remains true if the spaces $D([0,T],\Phi_{-p})$ and $D([0,T],\mathbb{R})$ are replaced by, respectively, $C([0,T],\Phi_{-p})$ and $C([0,T],\mathbb{R})$; see Mitoma

[22] (Proposition 4.1 and Remark R.1.). It may then be seen that our theorem III.2.1. also remains valid if the spaces $D([0,T],\Phi_{-p})$ and $D([0,T],\mathbb{R})$ are replaced by, respectively, $C([0,T],\Phi_{-p})$ and $C([0,T],\mathbb{R})$.
Since the basic ideas of the proof are unchanged by this substitution we omit the details.

—In order that theorem III.2.1. be applicable we need to be able to check whether $M^{n,T} \underset{n\to\infty}{\Longrightarrow} M^T$ on $D([0,T],\Phi_{-q_T})$. Corollary III.2.6 transforms this problem into a problem of checking weak convergence on $D([0,T],\mathbb{R})$ to which the classical results appearing, for example, in Billingsley's book [3] are applicable.

-Another often useful criterion for weak convergence on $D([0,T],\Phi_{-p})$ is the foolowing result by Mitoma ([22], theorem 5.3.2. and remark R.1):

<u>THEOREM B (MITOMA):</u>

Suppose that the sample paths of Y^n, $n \geq 1$ are in $D([0,T],\Phi_{-p})$ and that $(Y^n)_{n\geq 1}$ is uniformly k—continuous for some $k \geq p$. Suppose further that for every $\phi \in \Phi$ the sequence of distributions of $Y^n[\phi]$ is tight in $D([0,T],\mathbb{R})$ and for any finite number of elements $\phi_1,\ldots\phi_m \in \Phi$ and points $t_1,\ldots t_m \in [0,T]$ the distribution of $(Y^n_{t_1}[\phi],\ldots Y^n_{t_m}[\phi_m])$ converges in law as $n \to \infty$ to some m—dimensional probability distribution. Then there exists the limit process Y whose sample paths are in $D([0,T],\Phi_{-p})$ such that $Y^n \underset{n\to\infty}{\Longrightarrow} Y$.

CHAPTER IV

APPLICATIONS TO NEUROPHYSIOLOGICAL PROBLEMS

In this chapter we shall propose a new approach to modelling neuronal behaviour by means of Φ'–valued SDE's. We shall then employ the results of chapter III to giving three particular weak convergence results which are of interest for neuronal models.

Finally, we illustrate the application of our approach and results by giving a rigorous treatment and investigation of a model heuristically formulated and investigated by Wan and Tuckwell in [30]. But first we shall briefly describe the neurophysiological context. For a more detailed account hereof, we refer to [14] and the references therein. In our description we shall follow the introduction in [14].

A neuron is a cell whose principal function is to transmit information along its considerable length, which often exceeds one meter. "Information" is represented by changing amplitudes of electrical voltage potentials across the cell wall. A quiescent neuron will exhibit a resting potential of about 60 mV, the inside more negative than the outside. Under certain circumstances the voltage potential in the neuron dendrite will rise above a threshold point at which positive feedback causes a pulse of up to 100 mV to appear at the base of the dendrite; this pulse is transmitted rapidly along the body and down the axon of the cell until it reaches the so-called "pre-synaptic terminals" at the other end of the neuron. Here the pulse causes tiny vesicles filled with chemicals called "neurotransmitters" to empty into the narrow gaps between the presynaptic terminals and the dentrites of other neurons. When these chemicals diffuse across the gap and hit the neighboring neurons' dendrites, they may cause the voltage potential in these dentrites to rise above a threshold point and initiate another pulse.

Let $\xi(t,x)$ represent the difference between the voltage potential at time t at the location $x \in X$ (= surface of the neuron) and the resting potential of about -60 mV. As time passes, ξ evolves due to two separate causes:
(i) Diffusion and leaks: Depending on the nature of X, the electrical properties of the cell wall may be approximated by postulating a contraction semigroup $\{T_t\}$

on $L^2(X,\Gamma)$ where Γ is a suitable σ-finite measure on X. For example, if $X = [0,b]$, core conductor theory suggest the semigroup corresponding to the diffusion equation

$$\frac{\partial \xi}{\partial t} = -\beta \xi_t + \delta \Delta \xi_t \quad (\beta, \delta > 0)$$

with Neumann (or insulating) boundary conditions at both ends. In neural material like heart muscle in which electrical signals can travel more easily in some directions than in others, the Laplacian should be replaced by a more general second-order elliptic operator.

(ii) Random fluctuations: Every now and then a burst of neurotransmitter will hit some place or another on the membrane and suddenly the membrane potential will jump up or down by a random amount at a random time and location. It is believed that these random jumps are quite small and quite frequent, making it reasonable to hope that they can be modelled by a Gaussian noise process; in any case the arrivals at distant locations or in disjoint time intervals are believed to be aproximately independent, justifying their modelling as a mixture of Poisson processes or as a generalised Poisson process.

Because of the problem mentioned in chapter I that stochastic partial differential equations may not have a solution except in the form of a generalized process, we shall model the voltage potential ξ as a Φ'-valued process, where Φ is a countably Hilbert nuclear space.

In [14] Kallianpur and Wolpert used a Poisson process $N(A \times B \times (0,t])$ to represent the number of voltage pulses of size $a \in A$ arriving at sites $x \in B \subset \mathcal{X}$ (= surface of the neuron) at times prior to t.

Here, we adopt the point of view that, in practice, one can only "average" over the sites. Therefore it seems more realistic to assume that the arrival sites are given by "generalized functions" (distributions) $\eta \in \Lambda \subset \Phi'$, rather than by points x on the surface of the neuron membrane $\mathcal{X}$. As we shall see, this approach will also offer the advantage of enlarging the class of possible models.

To pursue this idea let us again consider a real rigged Hilbert space $\Phi \hookrightarrow H \hookrightarrow \Phi'$. Let $\mathcal{B}(\Phi')$ denote the Borel σ-field on Φ' and recall that $\mathcal{B}(\Phi')$ is the same whether we use the weakly or the strongly open sets in Φ' to define it.

Let $\Lambda \in \mathcal{B}(\Phi')$ and let, for each $n \in \mathbb{N}$, μ^n be a σ-finite positive measure on $(\mathbb{R} \times \Lambda, \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\Lambda))$ satisfying:

The mapping: $Q^n : \Phi \times \Phi \to \mathbb{R}$ defined by

$Q^n(\phi,\psi) = \int_{\mathbb{R} \times \Lambda} a^2 \eta[\phi]\eta[\psi]\mu^n(da\,d\eta)$ is continuous on $\Phi \times \Phi$.

Let N^n be a Poisson random measure on $(\mathbb{R}\times\Lambda\times[0,\infty);\mathcal{B}(\mathbb{R})\times\mathcal{B}(\Lambda)\times\mathcal{B}([0,\infty))$ with intensity measure $\mu^n(da\,d\eta)dt$ ($a \in \mathbb{R}$, $\eta \in \Lambda$, $t \in [0,\infty)$) (such a random measure exists, see e.g. Ikeda and Watanabe [9], page 42).

Let $\tilde{N}^n(da\,d\eta\,ds) = N^n(da\,d\eta\,ds) - \mu^n(da\,d\eta)ds$

and put

$$\tilde{Y}^n_t(\phi) = \int_{\mathbb{R}\times\Lambda\times[0,t]} a\eta[\phi]\tilde{N}^n(da\,d\eta\,ds);\ \phi \in \Phi.$$

Let $m^n \in \Phi'$, and define

$$\tilde{X}^n_t(\phi) = tm^n[\phi] + \tilde{Y}^n_t(\phi);\ \phi \in \Phi.$$

Then, for each $\phi \in \Phi$, $\tilde{X}^n_t(\phi)$ is a real CADLAG semimartingale satisfying

$$E(\tilde{X}^n_t(\phi))^2 = t^2 m^n[\phi]^2 + tQ^n(\phi,\phi).$$

Since Q^n is continuous on $\Phi\times\Phi$, the Kernel theorem for nuclear spaces (see Gelfand & Vilenkin [6], page 74) yields the existence of $r(n) \in \mathbb{N}$ and $C(n) > 0$ such that

$$m^n[\phi]^2 + Q^n(\phi,\phi) \leq C(n)\|\phi\|^2_{r(n)}, \quad \forall\ \phi \in \Phi.$$

We shall henceforth assume that the same r and C will do for all $n \in \mathbb{N}$, i.e. we suppose that there exists $r_2 \in \mathbb{N}$,

$C > 0$ such that

$$(1) \qquad m^n[\phi]^2 + Q^n(\phi,\phi) \le C\|\phi\|_{r_2}^2 \quad \forall\, n \in \mathbb{N} \quad \forall\, \phi \in \Phi.$$

Then, for any $T > 0$,

$$E\sup_{0 \le t \le T} (X_t^n(\phi))^2 \le 2C(4T + 2T^2)\|\phi\|_{r_2}^2 \quad \forall\, n \in \mathbb{N} \quad \forall\, \phi \in \Phi$$

and therefore Theorem III.1.12 and Remark 7 yields the existence of $q \in \mathbb{N}$, $q \ge r_2$ (independent of *n*) and a Φ_{-q}–valued CADLAG regularization X_t^n of $\{\tilde{X}_t^n(\phi) : \phi \in \Phi\}$. As usual, let $X_t^{n,T} := (X_t^n)_{t \in [0,T]}$; $T > 0$.

Let $m \in \Phi'$ and let $Q : \Phi \times \Phi \to \mathbb{R}$ be a continuous bilinear symmetric functional satisfying

$$(2) \qquad m[\phi]^2 + Q(\phi,\phi) \le C\|\phi\|_{r_2}^2.$$

A Φ'–valued Wiener process $W = (W_t)_{t \ge 0}$ with parameters m and Q is now defined precisely as in chapter II (see page 7) and Theorem II.1.1. (existence of Φ'–valued Wiener processess) remains true for the more general Φ considered here (with the understanding that the q in Theorem II.1.1. must now be replaced by $q_1 = \min\{r : i_r^{r_2}$ is Hilbert–Schmidt$\}$). Henceforth $W = (W_t)_{t \ge 0}$ shall denote a Φ'–valued Wiener process with parameters m and Q.

We may, and shall, choose $q \ge r_2$ such that

$X^{n,T} \in D([0,T],\Phi_{-q})$ P-a.s. $\forall\, n \in \mathbb{N}$ $\forall\, T > 0$ and

$W^T \in C([0,T],\Phi_{-q})$ P-a.s. $\forall\, T > 0$.

Let P_T^n denote the measure induced on $D([0,T],\Phi_{-q})$ by $X^{n,T}$ and let P_T denote the measure induced on $C([0,T],\Phi_{-q}) \subset D([0,T],\Phi_{-q})$ by W^T.

IV.1.1. THEOREM:

Suppose that, in addition to assumption (1),

(3) $$Q^n(\phi,\phi) \underset{n\to\infty}{\longrightarrow} Q(\phi,\phi) \quad \forall\, \phi \in \Phi.$$

(4) $$\lim_{n\to\infty} \int_{\mathbb{R}\times\Lambda} |a(\eta[\phi])|^3 \mu^n(da\,d\eta) = 0 \quad \forall\, \phi \in \Phi.$$

(5) $$m^n[\phi] \underset{n\to\infty}{\longrightarrow} m[\phi] \quad \forall\, \phi \in \Phi.$$

Then, for any $T > 0$, we have

$$P_T^n \underset{n\to\infty}{\Longrightarrow} P_T.$$

PROOF:

Fix $T > 0$. Let $t_1 \le t_2 \le \dots \le t_K \in [0,T]$ and $\phi_1,\dots,\phi_K \in \Phi$ for $K \in \mathbb{N}$ fixed.

We must show that

(i) $(X^n_{t_k}[\phi_k])_{k=1}^K$ converges in distribution to $(W_{t_k}[\phi_k])_{k=1}^K$

(ii) $\{P^n_T : n \in \mathbb{N}\}$ is tight on $D([0,T],\Phi_{-q})$.

<u>(i):</u> The log characteristic function of

$(X^n_{t_k}[\phi_k])_{k=1}^K$ is:

$$C(a_1,\ldots a_K) = \Big[i \sum_{k=1}^{K} t_k a_k m^n[\phi_k] +$$

$$\int_0^T \int_{\mathbb{R}\times\Lambda} (e^{ia\eta[F(s)]} - 1 - ia\eta[F(s)])\mu^n(da\,d\eta)ds \Big]$$

where $F(s) := \sum_{k=1}^{K} a_k 1_{[0,t_k]}(s)\phi_k$,

while that of $(W_{t_k}[\phi_k])_{k=1}^K$ is

$$C(a_1,\ldots a_K) = \Big[i \sum_{k=1}^{K} t_k a_k m[\phi_k] -$$

$$\frac{1}{2}\int_0^T Q(F(s),F(s))ds \Big].$$ Hence

$$|C_n(a_1,\ldots a_K) - C(a_1,\ldots,a_K)| =$$

$$\Big| i \sum_{k=1}^{K} t_k a_k (m^n[\phi_k] - m[\phi_k]) +$$

$$\int_0^T \Big[\int_{\mathbb{R}\times\Lambda} \sum_{p=3}^{\infty} (ia\eta[F(s)])^p \mu^n(da\,d\eta) - \frac{1}{2}(Q^n(F(s),F(s)) -$$

$$Q(F(s),F(s))) \Big] ds \Big| \leq$$

$$\Big| \sum_{k=1}^{K} t_k a_k (m^n[\phi_k] - m[\phi_k]) \Big| +$$

$$\int_0^T \int_{\mathbb{R}\times\Lambda} \Big| \sum_{p=3}^{\infty} (ia\eta[F(s)])^p \Big| \mu^n(da\,d\eta) ds +$$

$$\int_0^T \frac{1}{2} |Q^n(F(s),F(s)) - Q(F(s),F(s))| ds$$

$$\leq \sum_{k=1}^{K} t_k |a_k| |m^n[\phi_k]| +$$

$$\int_0^T \int_{\mathbb{R}\times\Lambda} |a\eta[F(s)]|^3 \mu^n(da\,d\eta) ds +$$

$$\int_0^T \frac{1}{2} |Q^n(F(s),F(s)) - Q(F(s),F(s))| ds$$

the first term tends to zero by (5). As for the second term, use (4) to obtain

$$(6) \qquad \lim_{n\to\infty} \int_{\mathbb{R}\times\Lambda} |a\eta[F(s)]|^3 \mu^n(da\,d\eta) = 0$$

$\forall\ s \in [0,T]$.

Now, by definition of F(s)

$$|a\eta[F(s)]| \le \sum_{k=1}^{K} |a||a_k||\eta[\psi_k]| \qquad \forall\ s \in [0,T].$$

Define $a_k^* := a_k \operatorname{sign}(a_k\eta[\psi_k])$. Then

$$\sum_{k=1}^{K} |a||a_k||\eta[\psi_k]| = |a| \sum_{k=1}^{K} a_k^*\eta[\psi_k]$$

$$= |a\eta[\sum_{k=1}^{K} a_k^*\psi_k]|$$

so

$$(7) \qquad |a\eta[F(s)]| \le |a\eta[\sum_{k=1}^{K} a_k^*\psi_k]| \qquad \forall\ s \in [0,T].$$

But $\sum_{k=1}^{K} a_k^*\psi_k \in \Phi$,

so an application of (4) gives

$$\lim_{n\to\infty} \int_{\mathbb{R}\times\Lambda} |a\eta[\sum_{k=1}^{K} a_k^*\psi_k]|^3 \mu^n(da\,d\eta) = 0$$

and thus

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}\times\Lambda} |a\eta[\sum_{k=1}^{K} a_k^* \phi_k]|^3 \mu^n(da\,d\eta) < \infty.$$

But then

$$\int_0^T \int_{\mathbb{R}\times\Lambda} |a\eta[F(s)]|^3 \mu^n(da\,d\eta)\,ds \xrightarrow[n\to\infty]{} 0$$

by (6), (7) and the DCT.

Further,

$$Q^n(F(s),F(s)) \xrightarrow[n\to\infty]{} Q(F(s),F(s))$$

for each $s \in [0,T]$ by (3) and since Q and Q^n satisfy (1) we have

$$|Q^n(F(s),F(s)) - Q(F(s),F(s))| \leq 2C\|F(s)\|^2_{r_2}.$$

Moreover, $\int_0^T \|F(s)\|^2_{r_2}\,ds < \infty$ so the DCT gives

$$\int_0^T |Q^n(F(s),F(s)) - Q(F(s),F(s))|\,ds \xrightarrow[n\to\infty]{} 0$$

concluding the proof of (i).

(ii) By Mitoma, [22], theorem 4.1 and remark R1, (see Theorem A, chapter III page 135) it is sufficient to show that

(a) $\forall\ \phi \in \Phi : \{X^{n,T}[\phi] : n \in \mathbb{N}\}$ is tight on $D([0,T],\mathbb{R})$

and

(b) $\exists\ k \geq q : \forall\ \epsilon > 0\ \forall\ \rho > 0\ \exists\ \delta > 0 :$

$$\|\phi\|_k < \delta \Rightarrow P(\sup_{0 \leq t \leq T} |X_t^n[\phi]| > \epsilon) < \rho \quad \forall\ n \in \mathbb{N}$$

For part (a), by Billingsley [3] theorem 15.3 page 125, it is sufficient to show that $\forall\ \phi \in \Phi$:

(ai) $\forall\ \eta > 0\ \exists\ a > 0 :$

$$P(\sup_{0 \leq t \leq T} |X_t^n[\phi]| > a) \leq \eta \quad \forall\ n \in \mathbb{N}$$

(aii) $\forall\ \epsilon > 0,\ \eta > 0\ \exists\ \delta \in (0,T)\ \exists\ n_o \in \mathbb{N}$:

$$P\{\sup_{t_1 \leq t \leq t_2} \min\{|X_t^n[\phi] - X_{t_1}^n[\phi]|, |X_{t_2}^n - X_t^n[\phi]|\} \geq \epsilon\} \leq \eta$$

$$\forall\ n \geq n_o$$

<u>and</u>

$$P(\sup_{s,t \in [0,\delta)} |X_s^n[\phi] - X_t^n[\phi]| \geq \epsilon) \leq \eta \quad \forall\ n \geq n_o$$

<u>and</u>

$$P(\sup_{s,t\in[T-\delta,T)} |X_s^n[\phi] - X_t^n[\phi]| \geq \epsilon) \leq \eta \quad \forall\, n \geq n_o$$

Fix $\phi \in \Phi$, and let $\eta > 0$, $\epsilon > 0$. Then,

$$P(\sup_{t\in[0,T]} |X_t^n[\phi]| > a) \leq \frac{1}{a^2} E(\sup_{t\in[0,T]} |X_t^n[\phi]|)^2$$

$$\leq \frac{2}{a^2} E(\sup_{t\in[0,T]} (t^2 m^n[\phi]^2 + \tilde{Y}_t^n[\phi]^2))$$

$$\leq \frac{2C}{a^2}(T^2 m^n[\phi]^2 + 4TQ^n(\phi,\phi))$$

$$\leq \frac{2}{a^2}(T^2 + 4T)\, C\|\phi\|_{r_2}^2 \quad \forall\, n \in \mathbb{N} \quad \text{(by 1)}$$

$$< \eta \text{ for } a^2 > \frac{2}{\eta}(T^2 + 4T)C\|\phi\|_{r_2}^2$$

Next, let $D:=\{t\geq 0:\ t_1 \leq t \leq t_2 \text{ and } t_2-t_1 \leq \delta\}$. Then

$$P\{\sup_{t\in D} \min\{|X_t^n[\phi] - X_{t_1}^n[\phi]|, |X_{t_2}^n[\phi] - X_t^n[\phi]|\} \geq \epsilon\}$$

$$\leq \epsilon^{-2} E\{\sup_{t\in D} \min\{|X_t^n[\phi] - X_{t_1}^n[\phi]|, |X_{t_2}^n[\phi] - X_t^n[\phi]|\}\}^2$$

$$\leq \epsilon^{-2} E\{\sup_{t\in D} \min\{|X_t^n[\phi] - X_{t_1}^n[\phi]|^2, |X_{t_2}^n[\phi] - X_t^n[\phi]|^2)$$

$$\leq \epsilon^{-2} \min(E\sup_{t\in D} |X_{t-t_1}^n[\phi]|^2, E\sup_{t\in D} |X_{t-t_1}^n[\phi]|^2)$$

$$\leq \frac{2}{\epsilon^2}\min\{\sup_{t\in D} ((t - t_1)^2 m^n[\phi]^2) + E\sup_{t\in D} |Y_{t-t_1}^n[\phi]|^2,$$

$$\sup_{t\in D}((t_2-t)^2 m^n[\phi]^2) + E\sup_{t\in D}|Y^n_{t_2-t}[\phi]|^2)$$

$$= \frac{2}{e^2}(\delta^2 m^n[\phi]^2 + 4\delta Q^n(\phi,\phi))$$

$$\leq \frac{2}{e^2}(\delta^2 + 4\delta)C\|\phi\|^2_{r_2} \quad \forall\ n \geq 1 \quad \text{(by (1))}$$

$$< \eta\ \forall\ n \in \mathbb{N} \text{ if } (\delta^2 + 4\delta) < (\frac{2}{e^2\eta}C\|\phi\|^2_{r_2})^{-1}.$$

Further,

$$P(\sup_{s,t\in[0,\delta)} |X^n_s[\phi] - X^n_t[\phi]| \geq e)$$

$$\leq \frac{1}{e^2}E(\sup_{s,t\in[0,} |X^n_s[\phi] - X^n_t[\phi])^2$$

$$\leq \frac{1}{e^2}2(\delta^2 m^n[\phi]^2 + 4\delta Q^n(\phi,\phi)) \leq \frac{2}{e^2}(\delta^2 + 4\delta)C\|\phi\|^2_{r_2} \quad \forall\ n \geq 1$$

$$\leq \eta\ \forall\ n \in \mathbb{N} \text{ if } \delta^2 + 4\delta \leq (\frac{2}{\eta e^2}C\|\phi\|^2_{r_2})^{-1}.$$

Similarly,

$$P(\sup_{s,t\in[T-\delta,T]} |X^n_s[\phi] - X^n_t[\phi]| \geq e)$$

$$\leq 2e^{-2}(\delta^2 m^n[\phi]^2 + 4\delta Q^n(\phi,\phi)) \leq 2e^{-2}(\delta^2 + 4\delta)C\|\phi\|^2_{r_2}$$

$$\forall\ n \geq 1$$

$$\leq \eta\ \forall\ n \in \mathbb{N} \text{ if } \delta^2 + 4\delta < (\frac{2}{e^2\eta}C\|\phi\|^2_{r_2})^{-1}.$$

Hence (ai) and (aii) are satisfied for

$\delta^2 + 4\delta < (2\eta^{-1}\epsilon^{-2}C\|\phi\|^2_{r_2})^{-1}$, and $n_o = 1$. This proves (a).

(b): Fix $\phi \in \Phi$ and let $\epsilon, \eta > 0$.

Then $P(\sup_{t\in[0,T]} |X^n_t[\phi]| \geq \epsilon)$

$\leq \epsilon^{-2}E(\sup_{t\in[0,T]} |X^n_t[\phi]|)^2$

$\leq \epsilon^{-2}2(T^2m^n[\phi]^2 + 4TQ^n(\phi,\phi))$

$\leq \epsilon^{-2}2(T^2 + 4T)C\|\phi\|^2_{r_2} \leq \epsilon^{-2}2(T^2 + 4T)C\|\phi\|^2_q$ (by (1))

$\leq \eta \vee n \geq 1$ if $\|\phi\|^2_q \leq \delta^2 = \dfrac{\eta\epsilon^2}{2(T^2+4T)C}$.

This completes the proof of theorem IV.1.1.

$\Vert$

REMARK:

Note that conditions (3), (4) and (5) were not used in the tightness part of the proof. Hence we have

IV.2. PROPOSITION:

Let Q^n, m^n satisfy (1). Then, for every $T \geq 0$, the family $\{P^n_T : n \in \mathbb{N}\}$ is tight on $D([0,T], \Phi_{-q})$.

Let $A : \Phi \rightarrow \Phi$ be a linear and continuous, and suppose that A and $\{T_t : t \geq 0\}$ satisfy assumption AS.1. in section III. For each $n \in \mathbb{N}$ let $\xi^n = (\xi^n_t)_{t\geq 0}$ denote the unique solution to

$$d\xi^n_t = A'\xi^n_t dt + dX^n_t$$

$$\xi^n_o = \xi_n$$

and let $\eta = (\eta_t)_{t\geq 0}$ denote the unique solution to

$$d\eta_t = A'\eta_t dt + dW_t$$

$$\eta_o = \eta^o$$

IV.1.3. THEOREM

Suppose that, in addition to (1),

(8) $\quad Q^n(\phi,\phi) \xrightarrow[n\rightarrow\infty]{} Q(\phi,\phi) \quad \forall\ \phi \in \Phi$

(9) $\quad \lim\limits_{n\rightarrow\infty} \int_{\mathbb{R}\times\Lambda} |a(\eta[\phi]|^3 \mu^n(da\,d\eta) = 0 \quad \forall\ \phi \in \Phi$

(10) $\quad \exists\ r \in \mathbb{N}: \sup\limits_{n} \max\{E\|\eta^o\|^2_{-r}, E\|\xi_n\|^2_{-2}\} < \infty$ and

$\xi_n \Rightarrow \eta^o$ on Φ_{-r} as $n \rightarrow \infty$.

(11) $\quad m^n[\phi] \xrightarrow[n\rightarrow\infty]{} m[\phi] \quad \forall\ \phi \in \Phi.$

Then, for any $T > 0 \; \exists p_T \in \mathbb{N}$:

$$\xi^{n,T} \underset{n\to\infty}{\Longrightarrow} \eta^T \text{ on } D([0,T],\Phi_{-p_T})$$

where $\xi^{n,T} = (\xi_t^n)_{t\in[0,T]}$ and

$\eta^T = (\eta_t)_{t\in[0,T]}$.

PROOF:

(1), (8), (9) and (10) imply that $X^{n,T} \underset{n\to\infty}{\Longrightarrow} W^T$ on $D([0,T],\Phi_{-q})$ $\forall \; T \geq 0$ by theorem IV.1.1.. Moreover, (1) implies condition (29) of AS.3 in chapter III while (10) supplies the remaining assumption of theorem III.2.1., from which the conclusion is therefore obtained.

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Next, we shall give conditions under which the processes $X^{n,T}$ will converge weakly on $D([0,T],\Phi_{-q})$ to a process X^T constructed from a Poisson random measure N on $\mathbb{R}\times\Lambda\times[0,\infty)$ in the same way as X^n was constructed from N^n. We shall then invoke theorem III.2.1. to give sufficient conditions for the weak convergence of $\xi^{n,T}$ on $D([0,T],\Phi_{-p_T})$ to the solution to the SDE driven by X.

Let $m \in \Phi'$ and let μ be a σ-finite measure on $(\mathbb{R}\times\Lambda,\mathcal{B}(\mathbb{R}\times\mathcal{B}(\Lambda))$ satisfying

(11a) $\quad m[\phi]^2 + B(\phi,\phi) \leq C\|\phi\|^2_{r_2} \qquad \forall\, \phi \in \Phi$

(11b) $\quad \int_{\mathbb{R}\times\Lambda} |e^{ia\eta[\phi]} - 1 - ia\eta[\phi]|\mu(dad\eta) < \infty$

where

$$B(\phi,\phi) := \int_{\mathbb{R}\times\Lambda} a^2\eta[\phi]^2\mu(dad\eta); \qquad \forall\, \phi \in \Phi.$$

Let N be a Poisson random measure on $(\mathbb{R}\times\Lambda\times[0,\infty), \mathcal{B}(\mathbb{R}\times\mathcal{B}(\Lambda)\times\mathcal{B}([0,\infty)))$ with intensity measure $\mu(dad\eta)dt$ ($a \in \mathbb{R}$, $\eta \in \Lambda$, $t \geq 0$).

Define

$$\tilde{Y}_t(\phi) = \int_{\mathbb{R}\times\Lambda\times[0,T]} a\eta[\phi](N(dad\eta ds) - \mu(dad\eta)ds);$$

$$t \geq 0;\ \phi \in \Phi,$$

and $\tilde{X}_t(\phi) = tm[\phi] + \tilde{Y}_t(\phi)$.

Since the r_2 required in (11a) is the same as that of (1), theorem III.1.12 (b) implies the existence of a Φ_{-q} valued regularization $X = (X_t)_{t\geq 0}$ of $\{\tilde{X}_t(\phi) : \phi \in \Phi\}$. For each $T > 0$ let R_T denote the measure induced on $D([0,T),\Phi_{-q})$ by $X^T = (X_t)_{t\in[0,T]}$. Then we have:

IV.1.4. THEOREM

Let m^n and μ^n satisfy (1). Let m,μ satisfy (11a,b) and suppose that

$$(12) \qquad \int_{\mathbb{R}\times\Lambda} (e^{ia\eta[\phi]} - 1 - ia\eta[\phi])\mu^n(da\,d\eta) \xrightarrow[n\to\infty]{} $$

$$\int_{\mathbb{R}\times\Lambda} (e^{ia\eta[\phi]} - 1 - ia\eta[\phi])\mu(da\,d\eta) \quad \forall\ \phi \in \Phi$$

$$(13) \qquad m^n[\phi] \to m[\phi] \quad \forall\ \phi \in \Phi.$$

Then, for every $T > 0$,

$P_T^n \Longrightarrow R_T$ as $n \to \infty$.

PROOF:

Fix $T > 0$. Since (1) is assumed to hold $\{P_T^n : n \geq 1\}$ is tight on $D([0,T],\Phi_{-q})$. Hence it suffices to show finite dimensional convergence:

Let $0 \leq t_1 \leq \ldots \leq t_K \leq T$ and $\phi_k \in \Phi$; $k = 1,\ldots,K$.

Then the characteristic functions for

$(X_{t_1}^{n,T}[\phi_1],\ldots,X_{t_K}^{n,T}[\phi_K])$ and

$(X^T_{t_1}[\phi_1],\ldots,X^T_{t_K}[\phi_K])$ are, respectively,

$C_n(a_1,\ldots,a_K) =$

$$\exp\Big[im^n[\sum_{k=1}^{K} t_k a_k \phi_k] +$$

$$\int_0^T \int_{R\times\Lambda} (e^{ia\eta[F(s)]} - 1 - ia\eta[F(s)])\mu^n(dad\eta)ds\Big]$$

and

$C(a_1,\ldots,a_k) =$

$$\exp\Big[im[\sum_{k=1}^{K} t_k a_k \phi_k] +$$

$$\int_0^T \int_{R\times\Lambda} (e^{ia\eta[F(s)]} - 1 - ia\eta[F(s)])\mu(dad\eta)ds\Big]$$

where

$$F(s) := \sum_{k=1}^{K} a_k 1_{[0,t_k]}(s)\phi_k.$$

By (13) it is enough to show that

$$\lim_{n\to\infty} \exp\Big[\int_0^T \int_{R\times\Lambda} (e^{ia\eta[F(s)]} - 1$$

$$- ia\eta[F(s)])\mu^n(dad\eta)ds\Big] =$$

$$\exp\Big[\int_0^T \int_{R\times\Lambda} (e^{ia\eta[F(s)]} - 1 - ia\eta[F(s)]\mu(dad\eta)ds\Big]$$

now, F(s) is piecewise constant, i.e. there are $0 = s_o <\ldots< s_M = T$ and $\phi_1,\ldots\phi_M \in \Phi$ such that

$$F(s) = \begin{cases} \phi_j \text{ if } s \in [s_{j-1}, s_j) & j = 1,\ldots,M-1 \\ \phi_M \text{ if } s \in [s_{M-1}, T] \end{cases}$$

Hence

$$(e^{ia\eta[F(S)]} - 1 - ia\eta[F(s)]) =$$

$$\sum_{j=1}^{M-1} (e^{ia\eta[\phi_j]} - 1 - ia\eta[\phi_j])1_{[s_{j-1},s_j)}(s) +$$
$$(e^{ia\eta[\phi_M]} - 1 - ia\eta[\phi_M])1_{[s_{M-1},T]}(s)$$

so

$$\int_0^T \int_{R\times\Lambda} (e^{ia\eta[F(s)]} - 1 - ia\eta[F(s)]\mu^n(dad\eta)ds =$$

$$\int_0^T \Big[\sum_{j=1}^{M-1} ({}_{R\times\Lambda} (e^{ia\eta[\phi_j]} - 1 - ia\eta[\phi_j])\mu^n(dad\eta)$$

$$1_{[s_{j-1}, s_j)}(s) + \int_{\mathbf{R}\times\Lambda} (e^{ia\eta[\phi_M]} - 1$$

$$- ia\eta[\phi_M])\mu^n(da\,d\eta)1_{[s_{M-1}, T]}(s) \Big] ds =$$

$$\sum_{j=1}^{M} \int_{\mathbf{R}\times\Lambda} (e^{ia\eta[\phi_j]} - 1 - ia\eta[\phi_j])\mu^n(da\,d\eta)(s_j - s_{j-1})$$

(by (12)

$$\xrightarrow[n\to\infty]{} \sum_{j=1}^{M} \int_{\mathbf{R}\times\Lambda} (e^{ia\eta[\phi_j]} - 1 - ia\eta[\phi_j])\mu(da\,d\eta)$$

$$(s_j - s_{j-1}) =$$

(Recall that $\int_{\mathbf{R}\times\Lambda} \ldots\mu(da\,d\eta)$ is finite by (11b))

$$\int_0^T \int_{\mathbf{R}\times\Lambda} (e^{ia\eta[F(s)]} - 1 - ia\eta[F(s)])\mu(da\,d\eta)ds,$$

concluding the proof.

$\|$

Let ξ^0 be a Φ'-valued random variable and let $\xi = (\xi_t)_{t\geq 0}$ denote the unique solution to the Φ'-valued SDE

$$d\xi_t = A'\xi_t dt + dX_t$$

$$\xi_o = \xi^o$$

IV.1.5. THEOREM

Let m^n and μ^n satisfy (1), let m,μ satisfy (11a,b) and suppose that (12) and (13) hold. Suppose further that

(14) $\exists\, r \in \mathbb{N} : \sup_n \max\{E\|\ _n\|^2_{-r}, E\|\ ^o\|^2_{-r}\} < \infty$

and that $\xi_n \underset{n\to\infty}{\Longrightarrow} \xi^o$ on Φ_{-r}.

Then, for any $T > 0$, $\exists\, p_T \in \mathbb{N}$:

$$\xi^{n,T} \underset{n\to\infty}{\Longrightarrow} \xi^T \text{ on } D([0,T],\Phi_{-p_T})$$

where

$$\xi^T = (\xi_t)_{t\in[0,T]}.$$

PROOF:

Let $T > 0$. Recall that $q \geq r_2$ is such that the canonical injection $\ _q^{r_2}$ is Hilbert-Schmidt from $\Phi_q \to \Phi_{r_2}$. Let $\{\phi_j : j \in \mathbb{N}\}$ be a CONS in Φ_q consisting of elements of Φ. Then note that

$$E \sup_{0\leq t\leq T} \|X_t^{n,T}\|^2_{-q} =$$

$$E(\sup_{0\le t\le T} \sum_{j=1}^{\infty} (X_t^{n,T}[\phi_j])^2) \le$$

$$E(\sum_{j=1}^{\infty} \sup_{0\le t\le T} (X_t^{n,T}[\phi_j])^2) =$$

$$\sum_{j=1}^{\infty} E\sup_{0\le t\le T} (\tilde{X}_t^{n}[\phi_j])^2 \le$$

$$\sum_{j=1}^{\infty} 2(T^2 m^n[\phi_j]^2 + 4TQ^n(\phi_j,\phi_j)) \le \quad \text{(by (1))}$$

$$\sum_{j=1}^{\infty} 2C(T^2 \vee 4T) \| \phi_j \|_{r_2}^2 =$$

$$2C(T^2 \vee 4T) \| {}^{r_2}_{q} \|_{HS}^2 \quad \forall\, n \in \mathbb{N},$$

(where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm) i.e. III.(29) of AS.3, chapter III is satisfied. Moreover, $X^{n,T}, X^T \in D([0,T],\Phi_{-q})$ (P-a.s.) by assumption and X^n and X are Φ'-valued (weak) L^2-semimartingales. By Theorem IV.1.4, (1), (11a,b), (12) and (13) imply that $X^{n,T} \underset{n\to\infty}{\Longrightarrow} X^T$ on $D([0,T],\Phi_{-q})$. Since also (14) is supposed to hold, the assumptions of Theorem III.2.1. are satisfied and the conclusion therefore follows from this theorem.

//

-Next we shall give conditions for the weak convergence of a sequence W^n of Φ'-valued Wiener processes to another Φ'-valued Wiener process W, and then employ these together with Theorem III.2.1. to give the corresponding weak convergence result for the solutions to the SDE's driven by W^n and W, respectively.

Let, for $n \in \mathbb{N}$, $m^n \in \Phi'$ and let $B^n : \Phi x \Phi \to \mathbb{R}$ be bilinear symmetric functionals satisfying (1). Let $W^n = (W_t^n)_{t \geq 0}$ denote the Φ'-valued Wiener process with parameters m^n and B^n. (1) and Remark 7, chapter III imply that $W_t^n \in \Phi_{-q}$ $\forall\ t \geq 0$, for some q which does not depend on $n \in \mathbb{N}$.

IV.1.6. THEOREM

Suppose that, in addition to satisfying (1), B^n and m^n satisfy

(15) $$B^n(\phi,\phi) \underset{n\to\infty}{\longrightarrow} Q(\phi,\phi) \quad \forall\ \phi \in \Phi$$

(16) $$m^n[\phi] \underset{n\to\infty}{\longrightarrow} m[\phi] \quad \forall\ \phi \in \Phi.$$

Then, for each T > 0, we have

$$W^{n,T} \underset{n\to\infty}{\Longrightarrow} W^T \text{ on } C([0,T],\Phi_{-q}),$$

where $W^{n,T} = (W_t^n)_{t\in[0,T]}$ and W_t is the Φ'-valued Wiener process introduced on page 153.

PROOF:

We must prove that

$\forall\ t > 0 \quad \{W^{n,T} : n \in \mathbb{N}\}$ is tight on $C([0,T],\Phi_{-q})$ and

$\forall\ 0 \le t_1 \le \dots \le t_N \le T\ \forall\ \phi_1,\dots,\phi_N \in \Phi$:

$$(W^{n,T}_{t_j}[\phi_j])_{j=1}^{N} \underset{n\to\infty}{\Longrightarrow} (W^{T}_{t_j}[\phi_j])_{j=1}^{N}.$$

The tightness part is proved in the same way as the tightness part of theorem IV.1.1.

Now, a calculation shows that

$$E \exp(i \sum_{j=1}^{N} a_j W^{n,T}_{t_j} [\phi_j]) =$$

$$\exp\left[i \sum_{j=1}^{N} t_j a_j m^n[\phi_j] - \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} t_j t_k a_j a_k B^n(\phi_j,\phi_k) \right]$$

$$\underset{n\to\infty}{\longrightarrow} \qquad \text{(by (15) and (16))}$$

$$\exp\left[i \sum_{j=1}^{N} t_j a_j m[\phi_j] - \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} t_j t_k a_j a_k B(\phi_j,\phi_k) \right]$$

$$= E \exp(i \sum_{j=1}^{N} a_j W^{T}_{t_j} [\phi_j]).$$

$\Downarrow$

Letting $\eta^n = (\eta^n_t)_{t \geq 0}$ denote the unique solution to the SDE on Φ':

$$d\eta^n_t = A'\eta^n_t dt + dW^n_t$$

$$\eta^n_o = \eta_n$$

and $\eta = (\eta_t)_{t \geq 0}$ be the Φ'-valued process introduced on page 163 we have

IV.1.7. THEOREM

Let, in addition to (1), B^n and m^n satisfy (15) and (16) of theorem IV.6, and suppose that η_n and η^o satisfy

(17) $\quad \exists\, r \in \mathbb{N} : \sup_n \max\{E\|\eta_n\|^2_{-r}, E\|\eta^o\|_{-r}\} < \infty$ and

$$\eta_n \underset{n\to\infty}{\Longrightarrow} \eta \text{ on } \Phi_{-r}.$$

Then, $\forall\, T > 0 \;\exists\, p_T \in \mathbb{N}$:

$$\eta^{n,T} \underset{n\to\infty}{\Longrightarrow} \eta^T \text{ on } C([0,T],\Phi_{-p_T}),$$

where $\eta^{n,T} := (\eta^n_t)_{t\in[0,T]}$.

PROOF:

By (1), (15) and (16) and theorem IV.1.6 $W^{n,T} \underset{n\to\infty}{\Longrightarrow} W^T$ on $D([0,T],\Phi_{-q})$ $\forall$ $t \geq 0$ where $q = \min\{p : i_p^{r_2}$ is Hilbert-Schmidt$\}$. Moreover (1) implies (29) of AS.3 in chapter III and (17) supplies the remaining condition of theorem III.2.1 (recall Remark 9 of Chapter III).

//

As indicated at the beginning of this section, Kallianpur and Wolpert ([14]) used Poisson random measures defined via intensity measures on $(\mathbb{R}\times\mathcal{X}, \mathcal{B}(\mathbb{R})\times\mathcal{B})$ where $(\mathcal{X}, \mathcal{B})$ is a suitable chosen measurable space, rather than by mean/covariance measures defined on $(\mathbb{R}\times\Lambda, \mathcal{B}(\mathbb{R})\times\mathcal{B}(\Lambda))$; $\Lambda \in \mathcal{B}(\Phi')$ as we have done it here.

It is therefore natural to address the question of when Kallianpur's and Wolpert's framework is contained in the one we have presented here. The following result gives a (partial) answer:

IV.1.8. PROPOSITION

Let $\mathcal{X}$ be a σ-compact topological Hausdorff space, and suppose that elements of ϕ are continuous functions on .
Further, suppose that

(18) $\{\delta_x : x \in \mathcal{X}\} \subset \Phi'$,

where, for each $x \in \mathcal{X}$, δ_x is the linear functional on Φ

given by

$$\delta_x[\phi] = \phi(x) \quad \forall\ \phi \in \Phi.$$

Then, for any $B \subset \mathcal{X}$ closed,

$$\{\delta_x : x \in B\} \in \mathcal{B}(\Phi').$$

REMARK

The conditions of the proposition are satisfied e.g. for $\mathcal{X} = \mathbb{R}^d$ and $\Phi = \mathcal{S}(\mathbb{R}^d)$.

Note also that a sufficient condition that (18) hold is that convergence in the Φ-topology implies pointwise convergence for functions on $\mathcal{X}$.

PROOF OF PROPOSITION IV.1.8:

By σ-compactness of $\mathcal{X}$, there exists a sequence $K_1 \subset K_2 \subset \ldots \subset K_n \subset \ldots$ of compact sets such that $\mathcal{X} = \bigcup_{n\geq 1} K_n$.

Let $B \subset \mathcal{X}$ be closed. Let $\Lambda_n = \{\delta_x : x \in B \cap K_n\}$.

Since $\{\delta_x : x \in B\} = \bigcup_{n\geq 1} \Lambda_n$ and $\mathcal{B}(\Phi')$ is generated by the weakly open sets in Φ', it suffices to show that Λ_n is weakly closed for all $n \in \mathbb{N}$:

Let $n \in \mathbb{N}$, and suppose that $(\delta_{x_\alpha})_{\alpha\in A}$ is a net in Λ_n converging weakly to some $\eta \in \Phi'$; i.e.

$$\delta_{x_\alpha}[\phi] \underset{\alpha}{\to} \eta[\phi] \quad \forall\, \phi \in \Phi.$$

Now, $\delta_{x_\alpha}[\phi] = \phi(x_\alpha) \quad \forall\, \phi \in \Phi.$

Since $x_\alpha \in K_n \cap B \;\forall\, \alpha \in A$ and $K_n \cap B$ is compact there is a subnet $\{x_\beta : \beta \in \Gamma\}$ which converges to x, say, in $K_n \cap B$.

Since each element of Φ is a continuous function on $\mathcal{X}$, it follows that

$$\delta_{x_\beta}[\phi] = \phi(x_\beta) \underset{\beta}{\to} \phi(x) \quad \forall\, \phi \in \Phi, \text{ i.e.}$$

$$\eta[\phi] = \lim_{\beta} \delta_{x_\beta}[\phi] = \phi(x) \quad \forall\, \phi \in \Phi.$$

Hence $\eta = \delta_x$, so $\eta \in \Lambda_n$ since $x \in K_n \cap B$, and therefore Λ_n is closed.

$\not\parallel$

Taking $B = \mathcal{X}$ in the proposition, we see that $\Lambda \in \mathcal{B}(\Phi')$, where $\Lambda := \{\delta_x : x \in \mathcal{X}\}$. Define a map

$$\Theta : \mathbb{R}\times\mathcal{X}\times[0,\infty) \to \mathbb{R}\times\Lambda\times[0,\infty) \text{ by}$$

$$\Theta(a,x,t) = (a,\delta_x,t).$$

It follows from the proposition that

Θ is $\mathcal{B}(\mathbb{R})\times\mathcal{B}(\Lambda)\times\mathcal{B}([0,\infty))/\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathcal{X})\times\mathcal{B}([0,\infty))$ measurable.

Similarly, the mapping τ: $\mathbb{R}\times\mathcal{X} \to \mathbb{R}\times\Lambda$ given by $\tau(a,x) = (a,\delta_x)$ is

$\mathcal{B}(\mathbb{R})\times\mathcal{B}(\Lambda)/\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathcal{X})$ measurable.

So if $m^n \in \Phi'$ and μ_1^n is a σ-finite measure on $\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathcal{X})$ satisfying

$m^n[\phi]^2 + Q_1^n(\phi,\phi) \leq C\|\phi\|_{r_1}^2 \quad \forall\ n \in \mathbb{N}$, where

$Q_1^n(\phi,\psi) := \int_{\mathbb{R}\times\mathcal{X}} a^2\phi(x)\psi(x)\mu_1^n(dadx)$; $\phi,\psi \in \Phi$

and $N_1^n(dadxdt)$ is a Poisson random measure on $\mathbb{R}\times\mathcal{X}\times[0,\infty)$ with intensity measure $\mu_1^n(dadx)dt$, then

$N^n(dad\eta dt) := N_1^n\ \Theta^{-1}$ is a Poisson random measure on $\mathbb{R}\times\Lambda\times[0,\infty)$ with intensity measure

$\mu^n(dad\eta)dt$, where $\mu^n = \mu_1^n \circ \tau^{-1}$,

and $Q^n(\phi,\psi) := \int_{\mathbb{R}\times\Lambda} a^2\eta[\phi]\eta[\psi]\mu^n(dad\eta)$

$$= \int_{\mathbb{R}\times\mathcal{X}} a^2\phi(x)\psi(x)\mu_1^n(dadx)$$

$$= Q_1^n(\phi,\psi) \quad \forall\ \phi,\psi \in \Phi.$$

So that Q^n together with m^n; $n \geq 1$ satisfy (1).

Therefore, under the conditions of proposition IV.1.8, the Kallianpur and Wolpert framework can indeed be represented in ours, and in this case their weak convergence result ([14], Theorem.3.2.) is analogue to our Theorem IV.1.3. [Recall from Proposition III.1.13 that the semigroup $\{T_t : t \geq 0\}$ with generator $-L$ considered in [14] satisfy our assumption AS.1 in section III.]. However, one would still have to verify the validity of the assumptions of Proposition IV.1.8. for each of the examples given in [14].

II.2.

Next, we shall apply our results to giving a rigorous formulation and investigation of a model recently proposed by Wan & Tuckwell [30]:

In order to study the behaviour of the difference $V(t,x)$ at time t between the so-called resting potential and the actual potential at point x on the surface of an infinitely thin cylinder shaped neuron which receives synaptic stimuli of the finite spatial extent ϵ_i at each of N sites x_i, Wan & Tuckwell investigated the model formally given by

$$(19)\quad \begin{cases} \dfrac{\partial V}{\partial t} = -V + \dfrac{\partial^2 V}{\partial x^2} + \displaystyle\sum_{i=1}^{N} h(x;x_i,\epsilon_i)(\alpha_i + \beta_i \dfrac{dW^i}{dt}) \\ V(0,x) = 0 \quad V(t,0) = 0 = V(t,b); \quad \forall\, t \geq 0, \end{cases}$$

where

$$h(x;x_i,\epsilon_i) = 1_{(x_i-\epsilon_i,x_i+\epsilon_i)}(x)$$

$(x_i, \epsilon_i > 0$ fixed for $i = 1,\ldots,N)$

and where W_t^i; $i = 1,\ldots,N$ are independent standard Wiener processes. α_i and β_i represent input current parameters and the neuron is thought of as the interval $[0,b]$; for some $b > 0$.

To see how this model can be given a rigorous representation as a Φ'-valued SDE, let $H = L^2([0,b])$ with inner product denoted by $\langle .,. \rangle_H$. Let L denote the operator $I - \Delta$ (Δ = Laplace operator in one dimension) with Neumann boundary conditions at 0 and b. Then L is a densely defined positive definite selfadjoint closed linear operator on H and admits a CONS $\{\phi_j : j = 0,1,2,\ldots\}$ in H consisting of eigenvectors of L;

$L\phi_j = \lambda_j\phi_j$; $j = 0,1,2,\ldots$, where $\lambda_j = 1 + \dfrac{j^2\pi^2}{b^2}$ and

$$\phi_j(x) = \begin{cases} b^{-1/2} \text{ if } j = 0 \\ \vdots \end{cases}$$

$$\left\{ \left(\frac{2}{b}\right)^{1/2} \operatorname{Cos}\left(\frac{j\pi x}{b}\right) \text{ if } j \geq 1. \right.$$

Further, $A := -L$ is the generator of a selfadjoint contraction semigroup $\{T_t : t \geq 0\}$ on H whose resolvent $R(\lambda) = (\lambda I - A)^{-1}$ is Hilbert-Schmidt on H.
Letting

$$\Phi := \{\phi \in H : \|(I - A)^r\phi\|_H < \infty \quad \forall\, r \in \mathbb{R}\}$$

and defining norms $\|\cdot\|_r$; $r \in \mathbb{R}$ on Φ by

$$\|\phi\|_r := \|(I - A)^r\phi\|_H;\ \phi \in \Phi$$

we put Φ_r equal to the $\|\cdot\|_r$-completion of Φ.

Then $\Phi = \bigcap_{r\in\mathbb{R}} \Phi_r$ and if τ denotes the Frechet topology on Φ generated by $\{\|\cdot\|_r : r \in \mathbb{R}\}$ (i.e. the projective limit topology on Φ)), then $(\Phi,\tau) \hookrightarrow H \hookrightarrow \Phi'$ (where Φ' denotes the strong dual of (Φ,τ)) is a rigged Hilbert space. Since $A = -L$, and L is a densely defined positive selfadjoint closed linear operator on H we see from Proposition III.1.13 that A and $\{T_t : t \geq 0\}$ satisfy AS.1 of chapter III.

Moreover, $\{\phi_j : j \in \mathbb{N}\} \subset \Phi$, $\Phi \subset \mathrm{Dom}(L)$ and per construction of Φ every element of Φ is an infinitely differentiable function. Let $N \in \mathbb{N}$ fixed, and for each $i = 1,\dots,N$ let $\xi_i \in \Phi'$. Let ν_i; $i = 1,\dots,N$ be ν-finite measures on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} a^2 \nu_i(da) < \infty \quad \forall\ i,$$

and let μ be the measure on $\mathbb{R}\times\Lambda$, where $\Lambda = \{\xi_i : i = 1,\dots,N\}$, given by

$$\mu = \sum_{i=1}^{N} \nu_i \times \delta_{\xi_i}; \text{ where } \delta_\xi \text{ is the point mass at } \xi.$$

Define

$$Q(\phi,\psi) = \int_{\mathbb{R}\times\Lambda} a^2 \eta[\phi]\eta[\psi]\mu(da\,d\eta);\ \phi,\psi \in \Phi$$

$$= \sum_{i=1}^{N} \int_{\mathbb{R}} a^2 \nu_i(da)\, \xi_i[\phi]\xi_i[\psi]$$

then Q is a continuous, bilinear symmetric functional on Φ, so for $m \in \Phi'$ given, let $W = W_t$ be the Φ'-valued (actually Φ_{-q} valued for some $q \in \mathbb{N}_o$; c.f. Theorem III.1.12) Wiener process with parameters m and Q.

Consider the SDE on Φ':

$$(20) \qquad d\eta_t = A'\eta_t dt + dW_t, \quad \eta_o = 0$$

Now, W is a weak Φ'-valued continuous L^2-semimartingale, and since A and $\{T_t : t \geq 0\}$ satisfy AS.1 there is a unique continuous Φ'-valued solution (from Theorem

III.1.5 and Remark 6) given by

$$\eta_t[\phi] = \int_0^t W_s[T_{t-s}A\phi]ds + W_t[\phi] \quad \forall\ \phi \in \Phi$$

(with probability one).

Choosing $\xi_i = \langle h(.;x_i,\epsilon_i),.\rangle_H \quad \forall\ i = 1,\dots,N$ and

$$m = m^{\epsilon} := \sum_{i=1}^{N} \alpha_i \xi_i, \ \beta_i^2 = \int_{\mathbb{R}} a^2 \nu_i(da),$$

(20) is the representation of (19) as an SDE on Φ'. To see that this is indeed the case, expand

$$\phi \in \sum_{j=0}^{\infty} \langle \phi,\phi_j\rangle_H\ \phi_j \text{ (converging in } (\Phi,\tau))$$

(recall that $\phi_j \in \Phi \quad \forall\ j \in \mathbb{N}$)

Then (writing η_t^{ϵ} for η_t and W_t^{ϵ} for W_t)

$$\eta_t^{\epsilon}[\phi] = \sum_{j=0}^{\infty} \left(\int_0^t W_s^{\epsilon}[T_{t-s}A\phi_j]ds + W_t^{\epsilon}[\phi_j]\right) \langle \phi,\phi_j\rangle_H$$

(converging in $L^2(\Omega,\mathbf{F},P)$).

Define for $x \in [0,b]$ and $n \in \mathbb{N}$

$$V_{\epsilon}^n(t,x) := \sum_{j=0}^{n} \left(\int_0^t W_s^{\epsilon}[T_{t-s}A\phi_j]ds + W_t^{\epsilon}[\phi_j]\right)\phi_j(x).$$

$$= \sum_{j=0}^{n} \left(\int_0^t -\lambda_j e^{-\lambda_j(t-s)} W_s^{\epsilon}[\phi_j]ds + W_t^{\epsilon}[\phi_j]\right)\phi_j(x)$$

Then, noting that $\sup_{x\in[0,b]} |\phi_j(x)| \le (\frac{2}{b})^{1/2}$ $\forall\ j \ge 0$ we find

$$(21) \qquad E \sum_{j=0}^{n} \sup_{0\le x\le b} \left| \int_0^t -\lambda_j e^{-\lambda_j(t-s)} W_s^{\epsilon}[\phi_j]ds + W_t^{\epsilon}[\phi_j]\right| |\phi_j(x)|$$

$$\le \sum_{j=0}^{n} E\left| \int_0^t -\lambda_j e^{-\lambda_j(t-s)} W_s^{\epsilon}[\phi_j]ds + W_t^{\epsilon}[\phi_j]\right| (\frac{2}{b})^{1/2}$$

and applying Itô's formula to the term inside each absolute value, we get

$$E \sum_{j=0}^{n} \sup_{0\le x\le b} \left| \int_0^t -\lambda_j e^{-\lambda_j(t-s)} W_s^{\epsilon}[\phi_j]ds + W_t^{\epsilon}[\phi_j]\right| |\phi_j(x)|$$

$$\le \sum_{j=0}^{n} E\left| \int_0^t e^{-\lambda_j(t-s)} dW_s^{\epsilon}[\phi_j]\right| (\frac{2}{b})^{1/2}$$

$$= \sum_{j=0}^{n} E\left| \int_0^t e^{-\lambda_j(t-s)} m_{\epsilon}[\phi_j]ds + \int_0^t e^{-\lambda j(t-s)} d\tilde{W}_s^{\epsilon}[\phi_j]\right| (\frac{2}{b})^{1/2}$$

(where $\tilde{W}_s^{\epsilon} := W_s^{\epsilon}[\phi_j] - sm_{\epsilon}[\phi_j]$; $j = 0,1,\ldots$)

$$\le (\frac{2}{b})^{1/2} \sum_{j=0}^{n} \Big[E(\int_0^t e^{-\lambda_j(t-s)} m_{\epsilon}[\phi_j]ds +$$

$$\int_o^t e^{-\lambda_j(t-s)} d\tilde{W}_s^{\epsilon}[\phi_j])^2\Big]^{1/2}$$

$$= (\frac{2}{b})^{1/2} \sum_{j=0}^{n} \Big[(\int_o^t e^{-\lambda_j(t-s)} m_{\epsilon}[\phi_j] ds)^2 +$$

$$\int_o^t e^{-2\lambda_j(t-s)} Q^{\epsilon}(\phi_j,\phi_j) ds\Big]^{1/2}$$

$$= (\frac{2}{b})^{1/2} \sum_{j=0}^{n} \Big[m_{\epsilon}[\phi_j]^2 \lambda_j^{-2} (1 - e^{-\lambda_j t})^2 +$$

$$\frac{1}{2\lambda_j}(1 - e^{-2\lambda_j t}) Q^{\epsilon}(\phi_j,\phi_j)\Big]^{1/2}$$

but $\lambda_j = 1 + \dfrac{\pi^2 j^2}{b^2}$

and $m_{\epsilon}[\phi_j] = \sum_{i=1}^{N} \alpha_i \langle h(.;x_i,\epsilon_i),\phi_j\rangle_H$

$$= \begin{cases} \sum_{i=1}^{N} \alpha_i \int_{x_i-\epsilon_i}^{x_i+\epsilon_i} (\frac{2}{b})^{1/2} \cos(\frac{j\pi x}{b}) dx & \text{for } j \geq 1 \\ \sum_{i=1}^{N} \alpha_i b^{-1/2} 2\epsilon_i & \text{for } j = 0 \end{cases}$$

so

$$|m_{\epsilon}[\phi_j]| \leq \begin{cases} 2(\frac{2}{b})^{1/2} \frac{b}{\pi j} \sum_{i=1}^{N} |\alpha_i| & \text{for } j \geq 1 \end{cases}$$

$$2b^{-1/2} \sum_{i=1}^{N} |\alpha_i \epsilon_i| \text{ for } j = 0$$

Also, for $j \geq 1$

$$Q^{\epsilon}(\phi_j,\phi_j) = \sum_{i=1}^{N} \beta_i^2 \left[\int_{x_i-\epsilon_i}^{x_i+\epsilon_i} \left(\frac{2}{b}\right)^{1/2} \cos\left(\frac{j\pi x}{b}\right) dx\right]^2$$

$$\leq \frac{8b}{\pi^2 j^2}\left(\sum_{j=1}^{N} \beta_i^2\right)$$

(recall that $\beta_i^2 := \int_{\mathbb{R}} a^2 \gamma_i(da)$)

while $Q^{\epsilon}(\phi_o,\phi_o) \leq \sum_{i=1}^{N} \frac{4\epsilon_i^2}{b} \beta_i^2$

so

$$\sum_{j=0}^{n} \left[m_{\epsilon}[\phi_j]^2 \lambda_j^{-2}(1 - e^{-\lambda_j t})^2 + \frac{1}{2\lambda_j}(1 - e^{-2\lambda_j t}) Q^{\epsilon}(\phi_j,\phi_j)\right]^{1/2}$$

$$\leq \text{CONSTANT} + \sum_{j=1}^{n} \left[\frac{8b}{\pi^2 j^2}\left(\sum_{i=1}^{N} |\alpha_i|\right)^2 \left(1 + \frac{\pi^2 j^2}{b^2}\right)^{-2} + \frac{1}{2}\left[1 + \frac{\pi^2 j^2}{b^2}\right]^{-1} \frac{8b}{\pi^2 j^2}\left(\sum_{i=1}^{N} \beta_i^2\right)\right]^{1/2}$$

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A COUNTABLY HILBE..(U) NORTH CAROLINA UNIV AT CHAPEL
HILL CENTER FOR STOCHASTIC PROC.. S K CHRISTENSEN
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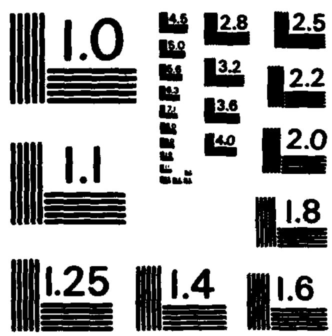

MICROCOPY RESOLUTION TEST CHART
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$< \infty \quad \forall\, n \in \mathbb{N},$

and combining this with (21) we see that the series

$$\sum_{j=0}^{\infty} \sup_{0\le x\le b} \left| \int_0^t - \lambda_j e^{-\lambda_j (t-s)} W^e[\phi_j]ds + W_t^e[\phi_j]\right| |\phi_j(x)|$$

is convergent (P—a.s.).

But then the sum defining $V_e^n(t,x)$ is absolutely convergent for all $x \in [0,b]$ P—a.s., and hence

$V_e(t,x) = \lim_{n\to\infty} V_e^n(t,x)$ exists for all $x \in [0,b]$ (P—a.s.) for each $t > 0$.

Moreover, there is a constant $C = C(t,\omega)$ such that

$$\sup_{n\in\mathbb{N}} \sup_{0\le x\le b} |V_e^n(t,x)| \le C \quad \text{(P—a.s.)} \quad \forall\, t \ge 0.$$

Therefore, the DCT gives

$$\langle V_e^n(t,.),\phi\rangle_H \underset{n\to\infty}{\longrightarrow} \langle V_e(t,.),\phi\rangle_H \quad \text{P-a.s.}$$

for each $t \ge 0$ and each $\phi \in \Phi$.

But $\langle V_e^n(t,.),\phi\rangle_H =$

$$\sum_{j=0}^{n} \left(\int_0^t W_s^e[T_{t-s}A\phi_j]ds + W_t^e[\phi_j] \right) \langle \phi, \phi_j \rangle_H$$

$$\xrightarrow[n\to\infty]{L^2(\Omega,\mathbf{F},P)} \eta_t^e[\phi]; \quad \phi \in \Phi, \text{ so}$$

$$\langle V_e(t,.), \phi \rangle_H = \eta_t^e[\phi], \quad P\text{-a.s.}$$

for each $\phi \in \Phi$ and $t \geq 0$.

To complete our argument that the process given by

$$V_e(t,x) = \sum_{j=0}^{\infty} \left[\int_0^t W_s^e[T_{t-s}A\phi_j]ds + W_s^e[\phi_j] \right] \phi_j(x)$$

$$\forall\ x \in [0,b] \quad P\text{-a.s.}$$

is the rigorous representation of the process formally given by (19), let us see that $EV_e(t,x)$ and $VarV_e(t,x)$ actually agree with the formulae found in [30] by a heuristic argument:

First we note that a simple computation will verify that, for each $x \in [0,b]$ and $t \geq 0$

$$V_e^n(t,x) \xrightarrow[n\to\infty]{L^2(\Omega,\mathbf{F},P)} V_e(t,x)$$

Therefore, we get

$$EV_{\epsilon}(t,x) = E\sum_{j=0}^{\infty}\left(\int_0^t W_s[T_{t-s}A\phi_j]ds + W_t[\phi_j]\right)\phi_j(x)$$

$$= E\sum_{j=0}^{\infty}\left(\int_0^t -\lambda_j e^{-\lambda_j(t-s)} W_s[\phi_j]ds + W_t[\phi_j]\right)\phi_j(x)$$

$$= E\sum_{j=0}^{\infty}\int_0^t e^{-\lambda_j(t-s)} dW_s[\phi_j]\phi_j(x)$$

$$= \sum_{j=0}^{\infty}\int_0^t e^{-\lambda_j(t-s)} m[\phi_j]ds\phi_j(x)$$

$$= \sum_{j=0}^{\infty} m[\phi_j]\lambda_j^{-1}(1 - e^{-\lambda_j t})\phi_j(x)$$

$$= \sum_{i=1}^{N}\alpha_i \sum_{j=0}^{\infty}\frac{\phi_j(x)\Phi_j(x_i;\epsilon_i)}{\lambda_j}(1 - e^{-\lambda_j t})$$

which is formula (8) page 279 in Wan & Tuckwell [30].

Here, as in [30],

$$\Phi_j(x_i;\epsilon_i) = \langle h(.;x_i,\epsilon_i),\phi_j\rangle_H$$

$$= \int_{x_i-\epsilon_i}^{x_i+\epsilon_i} \phi_j(x)dx.$$

Next,

$$\mathrm{Var}V_{\epsilon}(t,x) = E\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\int_0^t e^{-\lambda_j(t-s)} d\tilde{W}_s[\phi_j]$$

$$\int_0^t e^{-\lambda_k(t-s)}\, d\tilde{W}_s[\phi_k]\phi_j(x)\phi_k(x)$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\int_0^t e^{-(\lambda_j+\lambda_k)(t-s)}Q(\phi_j,\phi_k)ds\phi_j(x)\phi_k(x)$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{Q(\phi_j,\phi_k)}{\lambda_j+\lambda_k}\phi_j(x)\phi_k(x)(1 - e^{-(\lambda_j+\lambda_k)t})$$

$$= \sum_{i=1}^{N}\beta_i^2\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\phi_j(x)\phi_k(x)\psi_j(x_i;\epsilon_i)\psi_k(x_i;\epsilon_i)}{\lambda_j+\lambda_k}$$

$$(1 - e^{-(\lambda_j+\lambda_k)t})$$

which is formula (10) in [30].

Wan & Tuckwell proceed to compute the limit as $\epsilon_i \to 0$ $\forall\ i = 1,\dots,N$ in such a way that $\epsilon_i\alpha_i \to a_i$ and $\epsilon_i \;\;_i \to b_i > 0$ of $EV_\epsilon(t,x)$ and $VarV_\epsilon(t,x)$, and they find that these limits correspond to having point stimuli (i.e. $h(x,x_i,\epsilon_i)$ replaced by $\delta_{x_i}(x)$) at each of x_i; $i = 1,\dots,N$.

This result may be obtained from theorem IV.1.7 in the following manner:

For each $i = 1,\dots,N$ take $\nu_i = b_i\epsilon_i^{-1}\mu_i$; where μ_i a is finite measure on $\mathbb{R}$ with compact support.

Noting that every $\phi \in \Phi$ is a continuous function on $[0,b]$ (recall that $\Phi \subset \mathrm{Dom}(L)$ and that L is a differential operator) we let $\epsilon_i \to 0$ in such a way that $\epsilon_i \alpha_i \to a_i$. Then

$$\lim_{\epsilon_i \to 0} m_\epsilon[\phi] = \lim_{\epsilon_i \to 0} \sum_{i=1}^{N} \alpha_i \langle h(.;x_i,\epsilon_i),\phi\rangle_H$$

$$= \lim_{\epsilon_i \to 0} \sum_{i=1}^{N} \alpha_i \int_{x_i-\epsilon_i}^{x_i+\epsilon_i} \phi(x)dx$$

$$= \sum_{i=1}^{N} 2a_i \phi(x_i)$$

$$= \sum_{i=1}^{N} 2a_i \delta_{x_i}[\phi]$$

and

$$\lim_{\epsilon_i \to 0} Q^\epsilon(\phi,\phi) = \lim_{\epsilon_i \to 0} \sum_{i=1}^{N} \beta_i^2 \left(\int_{x_i-\epsilon_i}^{x_i+\epsilon_i} \phi(x)dx \right)^2$$

$$= \lim_{\epsilon_i \to 0} \sum_{i=1}^{N} b_i^2 \epsilon_i^{-2} \left(\int_{x_i-\epsilon_i}^{x_i+\epsilon_i} \phi(x)dx \right)^2 \int_{\mathbb{R}} a^2 \mu_i(da)$$

$$= \sum_{i=1}^{N} 4b_i^2 \phi(x_i)^2 \int_{\mathbb{R}} a^2 \mu_i(da)$$

$$= \sum_{i=1}^{N} 4b_i^2(\delta_{x_i}[\phi])^2 \cdot \int_{\mathbb{R}} a^2 \mu_i(da)$$

Also,

$$|m_\epsilon[\phi]|^2 + Q^\epsilon(\phi,\phi) \leq$$

$$\left[\left(\sum_{i=1}^{N} \|h(.;x_i,\epsilon_i)\|_H |\alpha_i|\right)^2 + \sum_{i=1}^{N} \beta_i^2 \|h(.,x_i,\epsilon_i)\|_H^2\right] \|\phi\|_H^2$$

$$= \left(\sum_{i=1}^{N} 2\epsilon_i |\alpha_i| + \sum_{i=1}^{N} 4b_i^2\right) \|\phi\|_H^2$$

$$\leq \text{CONSTANT} \ \|\phi\|_H^2 \quad \forall \ \epsilon_i,$$

since $\epsilon_i \alpha_i \to a_i$ and $\epsilon_i \to 0$; where CONSTANT is independent of ϵ_i, so condition (1) of section 1 is satisfied. Since the initial condition is zero, theorem IV.1.7 yields

$$\eta^{\epsilon,T} \underset{\epsilon_i \to 0}{\Longrightarrow} \eta^T \text{ on } C([0,T],\Phi_{-q_T}) \quad \forall \ T > 0$$

for some $q_T \geq 0$.

Here, $\eta = (\eta_t)_{t \geq 0}$ is the solution to (20) for

$$Q(\phi,\phi) = \sum_{i=1}^{N} 4b_i^2 \delta_{x_i}[\phi] \text{ and}$$

$$m[\phi] = \sum_{i=1}^{N} 2a_i \delta_{x_i}[\phi].$$

Now, take $\int_{\mathbb{R}} a^2 \mu_i(da) = 1$. Then

$$E\eta_t[\phi] = \sum_{i=1}^{N} 2a_i \sum_{j=0}^{\infty} \frac{\langle\phi,\phi_j\rangle_H \phi_j(x_i)}{\lambda_j}(1 - e^{-\lambda_j t}); \quad \phi \in \Phi$$

and

$$\operatorname{Var}\eta_t[\phi] = \sum_{i=1}^{N} 4b_i^2 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\langle\phi,\phi_j\rangle_H \langle\phi,\phi_k\rangle_H \phi_j(x_i)\phi_k(x_i)}{\lambda_j + \lambda_k}$$

$$(1 - e^{-(\lambda_j+\lambda_k)t}).$$

Since $V(t,x) = \sum_{j=0}^{\infty} \eta_t[\phi_j]\phi_j(x)$ (in $L^2(\Omega,\mathcal{F},P)$)

we get

$$\text{(22)} \qquad EV(t,x) = \sum_{i=1}^{N} 2a_i \sum_{j=0}^{\infty} \frac{\phi_j(x_i)}{\lambda_j}\phi_j(x)(1 - e^{-\lambda_j t})$$

and

$$\text{(23)} \qquad \operatorname{Var}V(t,x) = \sum_{i=1}^{N} 4b_i^2 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\phi_j(x_i)\phi_k(x_i)}{\lambda_j + \lambda_k}\phi_j(x)\phi_k(x).$$

(22) and (23) are the expressions found by Wan & Tuckwell

for point stimuli at $x_i; i = 1,\dots,N$.

In practice, equation (20) is likely to arise as a limit of equations where the noise is not a Wiener process, but rather a process generated by a Poisson random measure in the manner considered in section 1. As an illustration, take μ^n to be measures on $\mathbb{R}\times\Lambda$; where

$\Lambda = \{\xi_i : i = 1,\dots,N\}$ of the form

$$\mu^n = \sum_{i=1}^{N} \nu_i^n \times \delta_{\xi_i}, \text{ where}$$

for each $n \in \mathbb{N}$ and $i = 1,\dots,N$ ν_i^n is a σ-finite measure on $\mathbb{R}$ such that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} a^2 \nu_i^n(da) < C < \infty \quad \forall\, i = 1,\dots,N.$$

Let $m^n \in \Phi'$ converge weakly to m_e. hen there is $r \in \mathbb{N}_o$ such that

$$|m^n[\phi]|^2 \leq K\|\phi\|_r^2 \quad \forall\, n \in \mathbb{N}.$$

And since

$$|\xi_i[\phi]|^2 < (2e_i)^2\|\phi\|_o^2 \leq (2e_i)^2\|\phi\|_r^2$$

we get

$$|m^n[\phi]|^2 + Q^n(\phi,\phi) =$$

$$|m^n[\phi]|^2 + \sum_{i=1}^{N} \int_{\mathbb{R}} a^2 \nu_i^n(da)(\xi_i[\phi])^2 \le$$

CONSTANT $\|\phi\|_r^2 \quad \forall\ n \in \mathbb{N}$; i.e. (1) holds with $r_2 = r$.

Let X_i^n; $n \ge 1$ denote the Φ'-valued processes constructed from m^n and μ^n on p. 153.

Letting ξ_n denote the solution to

$$d\xi_t^n = -L' \ {}_t^n + dX_t^n$$

$$\xi_o^n = 0$$

Theorem IV.1.3 gives the existence of p_T such that

$$\xi^{n,T} \underset{n\to\infty}{\Longrightarrow} \eta^{\epsilon,T} \text{ on } D([0,T],\Phi_{-p_T})$$

provided that

(24) $\quad \lim\limits_{n\to\infty} \int_{\mathbb{R}} |a|^3 \nu_i^n(da) = 0 \quad \forall\ i = 1,\ldots,N$

and

(25) $\quad \lim\limits_{n\to\infty} \int_{\mathbb{R}} a^2 \nu_i^n(da) = \beta_i^2 \quad \forall\ i = 1,\ldots,N,$

i.e. the previously considered process η^{ϵ} can be thought of as the limit of solutions to SDE's with Poisson generated noise.

Physically, this type of weak convergence can be thought of as a situation in which the individual current stimuli of the neuron arrive very densely in each small time interval so as to create a total contribution to the electrical potential which behaves like the continuous Wiener process.

On the other hand, if (24) and (25) are replaced by

$$\lim_{n\to\infty} \int_{\mathbb{R}} (e^{iay} - 1 - iay)\nu_i^n(da) =$$

$$\int_{\mathbb{R}} (e^{iay} - 1 - iay)\nu_i^{\epsilon}(da) \quad \text{for all } y \in \mathbb{R}$$

then theorem IV.1.5 gives

$$\xi^{n,T} \underset{n\to\infty}{\Longrightarrow} \xi^{\epsilon,T} \text{ on } D([0,T],\Phi_{-p_T})$$

where ξ^{ϵ} is the process with mean functional m^{ϵ} constructed from the Poisson random measure with intensity

$$\mu^{\epsilon} = \sum_{i=1}^{N} \nu_i^{\epsilon} \times \delta_{\xi_i}.$$

This latter convergence can be thought of as modelling a situation in which the individual stimuli received by the neuron do not tend to arrive very densely packed in each small time interval, but rather tend to arrive clustered at random points of time.

-Let us conclude our discussion by briefly summarizing what we have obtained:

By proposing to represent the arrival sites of the stimuli of the neuron as distributions $\in \Phi'$ rather than by points x on the surface of the neuron we have given a rigorous representation (20) of the Wan & Tuckwell model (19) for the behaviour of the electrical potential in an infinitely thin neuron which receives stimuli of a spatial extent described by the distribution $\xi_i = \langle h(.;x_i,e_i),.\rangle_H$ at each of N points. -We wish to emphasize that it is not possible to incorporate the Wan & Tuckwell model into the framework used in [14].

We have then exhibited the solution as a Φ'-valued process η_t^e, with the interpretation that for suitable testfunctions ϕ (describing our measuring device) $\eta_t^e[\phi]$ represents the measured voltage potential difference at time t. We saw also that for the Wan & Tuckwell model the electrical potential $V^e(t,x)$ is well-defined at each point x of the surface of the neuron and $V^e(t,x)$ is related to η_t^e by

$$\eta_t^e[\phi] = \int_0^b V^e(t,x)\phi(x)dx \quad (P\text{-a.s.}) \quad \forall\ t \geq 0.$$

By means of Theorem III.2.1. (disguised as Theorem III.1.3) we then saw that η^{ϵ} can be thought of as the limit in distribution of processes driven by Poisson-generated stimuli, and further that (as was heuristically obtained by Wan & Tuckwell), as $\epsilon \to 0$ in an appropiate manner, $\eta^{\epsilon}_{\cdot}$ converges in distribution to the process $\eta_{\cdot}$, which describes the evolution of the electrical potential when stimulation occur precisely at the points $x_i; i = 1,\ldots,N$, of the neuronal surface.

Moreover, Theorem III.2.1. (in the form of Theorem IV.1.5) permitted us to give conditions under which the solution for Poisson generated stimuli would converge to a process still driven by Poisson generated stimuli.

It is our hope that we have hereby illustrated that the proposed approach of considering the arrival sites as given by distributions (rather than by points on the neuronal surface) together with Theorem III.2.1 and its consequences, provide a framework and a tool which is ample and powerful enough to permit the analysis of many aspects of the neuronal models.

For more general models of neuronal behaviour than (19), it may be of interest to estimate the mean functional m (which represents the mean arrival rate of stimuli) as well as testing hypothesis about m.
The results of chapter II should be useful in this

situation, which we hope to investigate in the future.

APPENDIX

Let us briefly recall the definition of a countably Hilbert nuclear space:

DEFINITION

Let Φ be a linear space upon which a sequence of real inner products $\langle .,. \rangle_n$; $n \in \mathbb{N}$ is given with the property that for all $n,m \in \mathbb{N}$ we have:
If $\{\phi_k\}_{k=1}^{\infty}$ Φ is a convergent sequence wrt.
$\|\cdot\|_n := \langle .,. \rangle_n^{1/2}$, and $\{\phi_k\}_{k=1}^{\infty}$ is Cauchy in $\|\cdot\|_m$, then $\{\phi_k\}_{k=1}^{\infty}$ is convergent in $\|\cdot\|_m$.
Let τ denote the Fréchet topology on Φ which is generated by the norms $\|\cdot\|_n$; $n \in \mathbb{N}$.
Then Φ is called a <u>countably</u> <u>Hilbert</u> space iff (Φ,τ) is complete.
(note that (Φ,τ) is metrizable by the metric d given by

$$d(\phi,\psi) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|\phi - \psi\|_n}{1 + \|\phi - \psi\|_n} ; \quad \phi,\psi \in \Phi.$$

(Φ,τ) is then complete iff (Φ,d) is complete.)

Let $(\Phi, \{\langle .,. \rangle_n : n \in \mathbb{N}\})$ be a countably Hilbert space and let $\|\cdot\|_n := \langle .,. \rangle_n^{1/2}$. Then we may, and shall henceforth, assume that

$$\|\phi\|_n \le \|\phi\|_m \quad \forall\, n \le m, \quad \forall\, \phi \in \Phi.$$

For each $n \in \mathbb{N}$ let Φ_n denote the completion of Φ wrt. $\|\cdot\|_n$. Then $\Phi_n \supset \Phi_m \quad \forall\, m \geq n$ and $\Phi = \bigcap_{n \geq 1} \Phi_n$.

<u>DEFINITION</u>

A countably Hilbert space Φ is called a countably Hilbert <u>nuclear</u> space iff we have

$\forall\, n \in \mathbb{N}\ \exists\, m \geq n$:
the canonical injection $i^n_m : \Phi_m \rightarrow \Phi_n$ is a Hilbert-Schmidt operator.

Let $\Phi_{-n} := \Phi'_n$ denote the strong dual of the Hilbert space Φ_n and let $\|\cdot\|_{-n}$ denote the Hilbert norm on Φ_{-n}. Let Φ' denote the strong topological dual of Φ, where Φ is a countably Hilbert nuclear space. Then

$\Phi' = \bigcup_{n \in \mathbb{N}} \Phi_{-n}$ with the (strict) inductive limit topology.

Moreover, on either Φ or Φ' a sequence is weakly convergent iff it is stongly convergent. The σ-field generated by the strongly open sets in Φ' is the same as that generated by the weakly open sets and it is therefore unambiguously called the Borel σ-field of Φ', and denoted $\mathcal{B}(\Phi')$. We refer to Gel'fand & Vilenkin [6], chapter 3 for the proof of these and other properties of countably Hilbert nuclear spaces.

DEFINITION

A triplet $\Phi \hookrightarrow H \hookrightarrow \Phi'$ where

(i) Φ is a countably Hilbert nuclear space and Φ' is the strong dual of Φ

(ii) H is the completion of Φ wrt. an inner product $\langle .,. \rangle_H$ on Φ which is continuous in the Φ-topology

is called a <u>rigged</u> Hilbert space.

A linear topological space can be a countably Hilbert nuclear space even if its topology at first appears to be generated by more than countably many seminorms:

Let Φ be a linear space upon which a family $\{\langle .,. \rangle_r : r \in \mathbb{R}\}$ of inner products are given with the property that

$$\|\phi\|_r \leq \|\phi\|_s \quad \forall\, r,s \in \mathbb{R} \quad \forall \phi \in \Phi;$$

where $\|\phi\|_r := (\langle \phi,\phi \rangle)^{1/2} \quad \forall\, \phi \in \Phi.$

Let $\emptyset \neq A \subset \mathbb{R}$ be any subset with the property that

(a) $\forall\, r \in \mathbb{R}\ \exists\, s \in A : s > r.$

Let $\mathcal{T}$ denote the Fréchet topology on Φ induced by $\{\|\cdot\|_r : r \in \mathbb{R}\}$ and let $\mathcal{T}_A$ denote the Fréchet topology on Φ induced by $\{\|\cdot\|_s : s \in A\}$.

THEOREM A.1

$$\bigcap_{r \in A} \Phi_r = \bigcap_{r \in \mathbb{R}} \Phi_r \quad \text{and} \quad \mathcal{T}_A = \mathcal{T},$$

where $\Phi_r := \|\cdot\|_r$ -completion of Φ.

Moreover, if

(b) $\quad \forall\, r \in \mathbb{N}\ \exists\, s \in \mathbb{N}$ with $s \geq 0$: the canonical injection $i_s^r : \Phi_s \to \Phi_r$ is Hilbert-Schmidt

and $\Phi = \bigcap_{r \in \mathbb{N}} \Phi_r$

then Φ is a countably Hilbert nuclear space and $\mathcal{T} = \mathcal{T}_{\mathbb{N}}$

PROOF:

Clearly, $\bigcap_{r \in \mathbb{R}} \Phi_r \subset \bigcap_{r \in A} \Phi_r$.

Conversely, let $\phi \in \bigcap_{r \in A} \Phi_r$. For a fixed $t \in \mathbb{R}$, pick $s \in A$, $s \geq t$. Since

$$\|\phi\|_t \leq \|\phi\|_s \quad \forall\, \phi \in \Phi \quad \text{we have } \Phi_s \subset \Phi_t.$$

But $\phi \in \bigcap_{r \in A} \Phi_r \Rightarrow \phi \in \Phi_s$, so $\phi \in \Phi_t$.

Since $t \in \mathbb{R}$ was arbitrary,

$$\bigcap_{r \in A} \Phi_r \subset \bigcap_{r \in \mathbb{R}} \Phi_r.$$

Next, the class of sets

$$C_A := \{ \{\phi \in \Phi : \|\phi\|_{r_i} < \epsilon_i, \ i=1,\dots,k\} : \\ k \in \mathbb{N}, \ r_i \in A \text{ and } \epsilon_i > 0 \quad \forall \ i=1,\dots,k\}$$

forms a complete neighbourhood base at zero for $\mathcal{T}_A$, while the class

$$C := \{ \{\phi \in \Phi : \|\phi\|_{r_i} < \epsilon_i, \ i=1,\dots,k\} : \\ k \in \mathbb{N}, \ r_i \in \mathbb{R}, \text{ and } \epsilon_i > 0 \quad \forall \ i=1,\dots,k\}$$

is a complete neighbourhood base at zero for $\mathcal{T}$. Let $F \in C$. Then

$F = \{\phi \in \Phi: \|\phi\|_{r_i} < \epsilon_i, \ i=1,\dots,k\}$ for some $k \in \mathbb{N}$, $\epsilon_i > 0$ and $r_i \in \mathbb{R}$.

By (a), for each $i=1,\dots,k$ we may choose $s_i \in A$ with $s_i > r_i$. Then, for every $\phi \in \Phi$:

$\|\phi\|_{r_i} \le \|\phi\|_{s_i}$, and hence

$$F \supset \{\phi \in \Phi : \|\phi\|_{s_i} < \epsilon_i \quad \forall\, i=1,\ldots,k\} \in C_A,$$

i.e. every τ-neighbourhood contains a τ_A-neighbourhood, i.e. every τ-open set is τ_A-open, so τ is weaker than τ_A. Conversely, $C_A \subset C$, so every τ_A-open set is τ-open, so τ_A is weaker than τ.

Finally, if $\Phi = \bigcap_{r \in \mathbb{N}} \Phi_r$ then Φ is necessarily complete, hence countably Hilbert, and therefore countably Hilbert nuclear by (b). $\tau = \tau_{\mathbb{N}}$ follows from the first part of the proof because $\mathbb{N}$ satisfies (a).

$\Vert$

- An important class of countably Hilbert nuclear spaces is constructed in the following manner:

Let $(H, \langle .,. \rangle_H)$ be a real separable Hilbert space and let L be a densely defined selfadjoint closed positive linear operator on H satisfying:

(c) $\quad \exists\, r_1 \in \mathbb{R}: \quad (\lambda I + L)^{-2r_1}$ is Hilbert-Schmidt on H

(c) implies that there is a CONS $\{\phi_j : j \in \mathbb{N}\}$ in H consisting of eigenvectors of L; $L\phi_j = \lambda_j \phi_j \quad \forall\, j \in \mathbb{N}$.
Define, for a fixed $\lambda > 0$,

$$\Phi = \{\phi \in H : \sum_{j=1}^{\infty} \langle \phi, \phi_j \rangle_H^2 \, (\lambda + \lambda_j)^{2r} < \infty \quad \forall\, r \in \mathbb{R}\}$$

i.e.

$$\Phi = \{\Phi \in H: \ \|(\lambda I + L)^r \phi\|_H^2 < \infty \quad \forall\, r \in \mathbb{R}\}$$

For each $r \in \mathbb{R}$ define an inner product $\langle .,. \rangle_r$ and a seminorm $\|.\|_r$ on Φ by

$$\langle \phi, \psi \rangle_r := \sum_{j=1}^{\infty} \langle \phi, \phi_j \rangle_H \langle \psi, \phi_j \rangle_H (\lambda + \lambda_j)^{2r}$$

and

$$\|\phi\|_r := \langle \phi, \phi \rangle_r^{1/2} ; \quad \phi, \psi \in \Phi.$$

Let Φ_r denote the $\|.\|_r$ -completion of Φ and give Φ the Frèchet topology induced by $\{\|.\|_r : r \in \mathbb{R}\}$. Letting Φ' denote the strong topological dual of Φ we have

(i) $\Phi = \bigcap_{r \in \mathbb{R}} \Phi_r ; \quad \Phi' = \bigcup_{r \in \mathbb{R}} \Phi_r$ with the inductive limit topology.

(ii) $\|\phi\|_r \leq \|\phi\|_s \quad \forall\, \phi \in \Phi$ and consequently $\Phi_r \supset \Phi_s$ $\forall\, r \leq s$.

(iii) $\forall\, r \in \mathbb{R}$: The canonical injection i_s^r : $\Phi_s \to \Phi_r$ is Hilbert-Schmidt for every $s \geq r + r_1$.

(iv) For $r \geq 0$ Φ_{-r} and Φ_r are in duality under the pairing

$$\eta[\phi] = \sum_{j=1}^{\infty} \langle \eta, \phi_j \rangle_{-r} \langle \phi, \phi_j \rangle_r \;;\; \eta \in \Phi_{-r} \quad \phi \in \Phi.$$

(v) $\Phi_o = H$.

(vi) $\{\phi_j : j \in \mathbb{N}\}$ is a complete orthogonal system in Φ_r for every $r \in \mathbb{R}$ with $\|\phi_j\| = (\lambda + \lambda_j)^r$ for each $j \in \mathbb{N}$.

(vii) $\sum_{j=1}^{\infty} (\lambda + \lambda_j)^{-2r_1} < \infty$.

Theorem A1 together with (i), (ii) and (iii) imply that Φ is a countably Hilbert nuclear space. We shall say that Φ is generated by $(\lambda I + L)$.

- The Schwartz space of all rapidly decreasing functions on $\mathbb{R}^d$ is generated by $(1/2I + L)$, where

$L = \frac{|x|^2}{4} - \triangle$. -See K. Ito [10] for details.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. A Φ'-valued map on Ω, which is $\mathcal{B}(\Phi')/\mathcal{F}$ -measurable is called a Φ'-valued random variable. A Φ'-valued map η: $I \times \Omega \to \Phi'$ where $I \quad \mathbb{R}$ is called a (stochastic) process iff $\eta_t : \Omega \to \Phi'$ is a Φ'-valued random variable for every $t \in I$.

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